澳門大學
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澳門理工學院
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Institute for Tourism Studies

澳門科技大學
UNIVERSIDADE DE CIÊNCIA E TECNOLOGIA DE MACAU
MACAU UNIVERSITY OF SCIENCE AND TECHNOLOGY

澳門四高校聯合入學考試（語言科及數學科）

Joint Admission Examination for Macao Four Higher Education Institutions (Languages and Mathematics)

2018 年試題及參考答案

2018 Examination Paper and Suggested Answer

數學附加卷　Mathematics Supplementary Paper

注意事項：

1. 考生獲發文件如下:

   1.1 本考卷包括封面共 22 版

   1.2 草稿紙一張

2. 請於本考卷封面填寫聯考編號、考場、樓宇、考室及座號。
3. 本考卷共有五條解答題，每題二十分，任擇三題作答。全卷滿分為六十分。
4. 必須在考卷內提供的橫間頁內作答，寫在其他地方的答案將不會獲評分。
5. 必須將解題步驟清楚寫出。只當答案和所有步驟正確而清楚地表示出來，考生方可獲得滿分。
6. 本考卷的圖形並非按比例繪畫。
7. 考試中不可使用任何形式的計算機。
8. 請用藍色或黑色原子筆作答。
9. 考試完畢，考生須交回本考卷及草稿紙。

Instructions:

1. Each candidate is provided with the following documents:

   1.1 Question paper including cover page – 22 pages

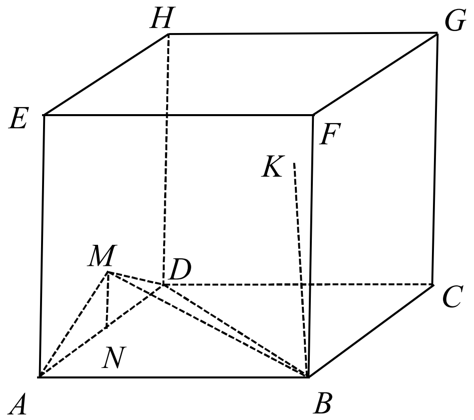
   1.2 One sheet of draft paper

2. Fill in your JAE No., campus, building, room and seat no. on the front page of the examination paper.
3. There are 5 questions in this paper, each carries 20 marks. Answer any 3 questions. Full mark of this paper is 60.
4. Put your answers in the lined pages provided. Answers put elsewhere will not be marked.
5. Show all your steps in getting to the answer. Full credits will be given only if the answer and all the steps are correct and clearly shown.
6. The diagrams in this examination paper are not drawn to scale.
7. Calculators of any kind are not allowed in the examination.
8. Answer the questions with a blue or black ball pen.
9. Candidates must return the question paper and draft paper at the end of the examination.

任擇三題作答，每題二十分。請把答案寫在緊接每條題目之後的 3 頁橫間頁內。

Answer any 3 questions, each carries 20 marks. Write down the answers on the 3 lined pages following each question.

1.



如上圖所示，*ABCD-EFGH* 是邊長為 4 的正立方體，*MN* 為 *ADHE* 中的線段，垂直平分 *AD* 於點 *N*，且 $|MN| = 1$，*K* 為正方形 *DCGH* 的中點。

(a) 求 $\cos\angle MBK$。 (10 分)

(b) (i) 證明 *MA* 垂直 *AB*。 (2 分)

(ii) 求三棱錐 *M-ABD* 的體積，從而求點 *D* 至平面 *AMB* 的距離。 (8 分)

In the above figure, *ABCD-EFGH* is a cube with edge length 4, *MN* is a segment in *ADHE* perpendicularly bisects *AD* at point *N*, and $|MN| = 1$. Point *K* is the center of the square *DCGH*.

(a) Find $\cos\angle MBK$. (10 marks)

(b) (i) Show that *MA* and *AB* are perpendicular. (2 marks)

(ii) Find the volume of the triangular pyramid *M-ABD*, and hence find the distance from point *D* to the plane *AMB*. (8 marks)

2. (a) 已知函數 $f(x)=x^3-3x^2-9x+2$ 。

(i) 求 $f'(x)$ 及 $f''(x)$ 。 (2 分)

(ii) 求 $f(x)$ 的局部極大點、局部極小點和拐點。 (6 分)

(iii) 繪出曲線 $y=f(x)$ 。 (2 分)

(iv) 繪出曲線 $y=f(|x|)-2$ 。 (2 分)

(b) 求由曲線 $y=x^3-3x^2-9x+2$ 和直線 $y=x+2$ 所包圍的區域的面積。 (8 分)

(a) Given function $f(x)=x^3-3x^2-9x+2$.

(i) Find $f'(x)$ and $f''(x)$. (2 marks)

(ii) Find the local maximum points, local minimum points and inflection points of $f(x)$. (6 marks)

(iii) Sketch the curve $y=f(x)$. (2 marks)

(iv) Sketch the curve $y=f(|x|)-2$. (2 marks)

(b) Find the area of the region bounded by the curve $y=x^3-3x^2-9x+2$ and the straight line $y=x+2$. (8 marks)

3. 已知拋物線 $P: y=(x-1)^2$ 及點 $A(-1,3)$ 。

(a) 設過點 A 且與 P 相切的兩條切線與 P 分別相切於點 B 及 C，求這兩條切線的方程，並求點 B 及 C 的座標。

[提示: 設 $y-3=m(x+1)$ 為切線的方程。] (12 分)

(b) 若 α 為這兩條切線的夾角，證明 $\tan\alpha=\frac{4}{13}$ 。 (2 分)

(c) 求三角形 ABC 的面積。 (6 分)

Given parabola $P: y=(x-1)^2$ and point $A(-1,3)$.

(a) Suppose the two lines passing through A and tangent to P touch P at points B and C.

Find the equations of the two tangent lines, and find the coordinates of points B and C.

[Hint. Suppose the equation of the tangent line is $y-3=m(x+1)$.] (12 marks)

(b) If α is the angle between the two tangent lines, show that $\tan\alpha=\frac{4}{13}$. (2 marks)

(c) Find the area of the triangle ABC. (6 marks)

4. (a) 求 $\dfrac{1}{x^2-1}$ 的部分分式，從而求 $\displaystyle\sum_{n=2}^{10}\frac{1}{n^2-1}$ 。 (6 分)

(b) 設 $\{a_n\}_{n=1}^{\infty}$ 為一數列，滿足 $a_1 = 2$ 及 $a_{n+1} = a_n + \dfrac{1}{a_n}$ ，$n = 1,2,3,\ldots$ 。

(i) 設 h 和 k 為正數，且 $h > \sqrt{k}$ 。證明 $h + \dfrac{1}{h} > \sqrt{k+2}$ 。

[提示: 考慮 $(h + \dfrac{1}{h})^2$ 。] (3 分)

(ii) 用 (i) 及數學歸納法，證明對任意正整數 n，$a_n > \sqrt{2n+1}$ 。 (3 分)

(iii) 設 $b_n = \dfrac{a_n}{\sqrt{n}}$ ，$n = 1,2,3,\ldots$ 。證明 $\dfrac{b_{n+1}}{b_n} < 1$ ，$n = 1,2,3,\ldots$ 。 (8 分)

(a) Find the partial fractions of $\dfrac{1}{x^2-1}$. Hence find $\displaystyle\sum_{n=2}^{10}\frac{1}{n^2-1}$. (6 marks)

(b) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence satisfying $a_1 = 2$ and $a_{n+1} = a_n + \dfrac{1}{a_n}$, $n = 1,2,3,\ldots$.

(i) Suppose h and k are positive numbers and $h > \sqrt{k}$. Show that $h + \dfrac{1}{h} > \sqrt{k+2}$.

[Hint: Consider $(h + \dfrac{1}{h})^2$.] (3 marks)

(ii) Using (i) and mathematical induction, show that for any positive integer n,

$$a_n > \sqrt{2n+1} .$$ (3 marks)

(iii) Let $b_n = \dfrac{a_n}{\sqrt{n}}$, $n = 1,2,3,\ldots$. Show that $\dfrac{b_{n+1}}{b_n} < 1$, $n = 1,2,3,\ldots$. (8 marks)

5. (a) 因式分解行列式 $\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$。 (8 分)

(b) (i) 求下列方程組的通解:

$$\begin{cases} x-2y+z=-3 \\ -2x+5y+z=5 \end{cases}$$ 。 (4 分)

(ii) 利用 (i) 的通解，或用其他方法，求常數 p 和 q 的值使得下列方程組有解，並給出方程組的解:

$$\begin{cases} x-2y+z=-3 \\ -2x+5y+z=5 \\ px-\ \ y-z=q \end{cases}$$ 。 (8 分)

(a) Factorize the determinant $\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$. (8 marks)

(b) (i) Find the general solution of the following system of equations:

$$\begin{cases} x-2y+z=-3 \\ -2x+5y+z=5 \end{cases}.$$ (4 marks)

(ii) Using the general solution obtained in (i), or otherwise, find the values of the constants p and q such that the following system of equations has a solution, and give the solution:

$$\begin{cases} x-2y+z=-3 \\ -2x+5y+z=5 \\ px-\ \ y-z=q \end{cases}.$$ (8 marks)

全卷完

End of Paper

參考答案：

1. (a) 因 $|BK|^2=24$，$|BM|^2=21$ 及 $|MK|^2=9$，故

$$\cos\angle MBK=\frac{|BM|^2+|BK|^2-|MK|^2}{2|BM||BK|}=\frac{3}{\sqrt{14}}$$ 。

(b) (i) 因 $BA\perp AEHD$ 及 $MA\subset AEHD$，故 $BA\perp MA$。

(ii) $M-ABD$ 的體積 $=\frac{1}{3}|MN|\left(\frac{1}{2}|AB||AD|\right)=\frac{8}{3}$ 。

設 h 為點 D 至平面 AMB 的距離，則

$D-AMB$ 的體積 $=\frac{1}{3}h\left(\frac{1}{2}|AM||AB|\right)=\frac{2\sqrt{5}h}{3}$，故 $h=\frac{4}{\sqrt{5}}$ 。

2. (a) (i) $f'(x)=3x^2-6x-9$ 及 $f''(x)=6x-6$ 。

(ii) $f'(x)=0\Leftrightarrow 3x^2-6x-9=0\Leftrightarrow 3(x-3)(x+1)=0\Leftrightarrow x=3$ or $x=-1$。

當 $x<-1$ 時，$f'(x)>0$， 故 $f(x)$ 是遞增的。

當 $-1<x<3$ 時，$f'(x)<0$， 故 $f(x)$ 是遞減的。

當 $3<x$ 時，$f'(x)>0$， 故 $f(x)$ 是遞增的。

因此，$f(-1)=7$ 是一局部極大點及 $f(3)=-25$ 是一局部極小點。

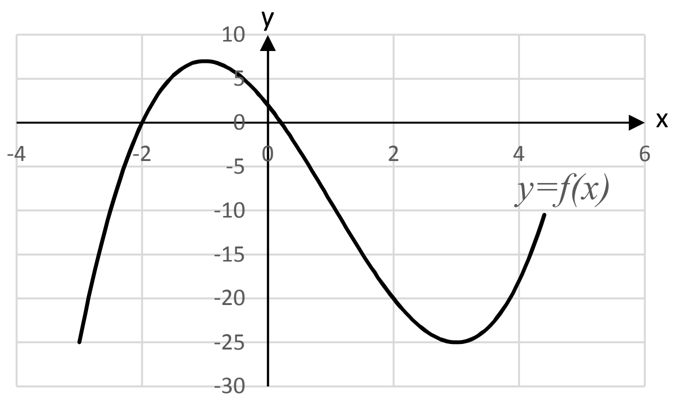
$f''(x)=0\Leftrightarrow 6x-6=0\Leftrightarrow x=1$ 。

當 $x<1$ 時，$f''(x)<0$，故 $f(x)$ 是凹的。

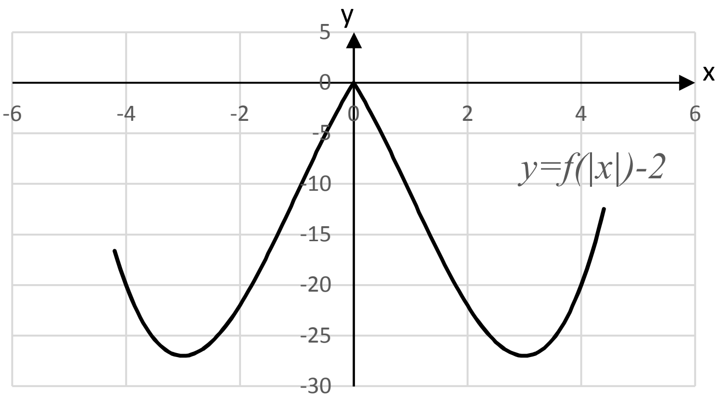
當 $x>1$ 時，$f''(x)>0$，故 $f(x)$ 是凸的。

因此，$f(1)=-9$ 是一拐點。

(iii)



(iv)



(b) 解 $\begin{cases} y = x^3 - 3x^2 - 9x + 2 \\ y = x + 2 \end{cases}$，得知曲線 $C: y = x^3 - 3x^2 - 9x + 2$ 和直線

$L: y = x + 2$ 交於 $x = -2$，$x = 0$ 和 $x = 5$。當 $-2 < x < 0$，曲線 C 是在直線 L 之上；當 $0 < x < 5$，曲線 C 是在直線 L 之下。故所求面積為

$$\int_{-2}^{0} (x^3 - 3x^2 - 9x + 2) - (x + 2)\, dx + \int_{0}^{5} (x + 2) - (x^3 - 3x^2 - 9x + 2)\, dx$$
$$= \int_{-2}^{0} x^3 - 3x^2 - 10x\, dx + \int_{0}^{5} -x^3 + 3x^2 + 10x\, dx$$
$$= \left[\frac{x^4}{4} - x^3 - 5x^2\right]_{-2}^{0} + \left[-\frac{x^4}{4} + x^3 + 5x^2\right]_{0}^{5}$$
$$= 8 + \frac{375}{4}$$
$$= \frac{407}{4}$$

3. (a) 設切線 L 的方程為 $y = m(x+1) + 3$。由 $\begin{cases} y = (x-1)^2 \\ y = m(x+1) + 3 \end{cases}$，得

$(x-1)^2 = m(x+1) + 3$，從而有

$$x^2 - (2+m)x - (2+m) = 0 \text{。} \qquad (*)$$

因 L 是 P 的切線，故 (*) 有二重根，其判別式等於 0。因此得

$[-(2+m)]^2 - 4(1)(-2-m) = 0$，即 $m^2 + 8m + 12 = 0$。因此得 $m = -6$ 或 $m = -2$。

設 $m = -6$。由 (*) 得 $x^2 + 4x + 4 = 0$，從而得 $x = -2$。此時，切線 L 的方程為

$y = -6x - 3$，由 $x = -2$ 得 $y = 9$。故切點為 $(-2, 9)$。

設 $m = -2$。由 (*) 得 $x^2 = 0$，從而得 $x = 0$。此時，切線 L 的方程為

$y = -2x + 1$，由 $x = 0$ 得 $y = 1$。故切點為 $(0, 1)$。

(b) 設 $\tan\theta_1 = -2$ 及 $\tan\theta_2 = -6$，其中 $-\frac{\pi}{2} < \theta_2 < \theta_1 < 0$，則 $\alpha = \theta_1 - \theta_2$。

因此 $\tan\alpha = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2} = \frac{(-2)-(-6)}{1+(-2)(-6)} = \frac{4}{13}$。

(c) 設 $B = (-2,9)$ 及 $C = (0,1)$。計算 $\tan\alpha = \frac{4}{13} \Rightarrow \sin\alpha = \frac{4}{\sqrt{4^2+13^2}} = \frac{4}{\sqrt{185}}$，

$|AB| = \sqrt{[-2-(-1)]^2 + (9-3)^2} = \sqrt{37}$，$|AC| = \sqrt{[0-(-1)]^2 + (1-3)^2} = \sqrt{5}$。

三角形 ABC 的面積為 $\frac{1}{2}(|AB|\sin\alpha)|AC| = 2$。

4. (a) 設 $\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$。從 $1 = A(x+1) + B(x-1) = (A+B)x + (A-B)$ 得 $\begin{cases} A+B=0 \\ A-B=1 \end{cases}$，

故 $A = \frac{1}{2}$ 及 $B = -\frac{1}{2}$。因此

$$
\begin{aligned}
\sum_{n=2}^{10} \frac{1}{n^2-1} &= \sum_{n=2}^{10}\left(\frac{1/2}{n-1} - \frac{1/2}{n+1}\right) = \sum_{n=2}^{10}\frac{1/2}{n-1} - \sum_{n=2}^{10}\frac{1/2}{n+1} \\
&= \frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{9}\right) - \frac{1}{2}\left(\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{11}\right) = \frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{10}-\frac{1}{11}\right) \\
&= \frac{36}{55}
\end{aligned}
$$

(b)(i) 因 $\left(h+\frac{1}{h}\right)^2 = h^2 + 2 + \frac{1}{h^2} > k + 2 + \frac{1}{h^2} > k+2$，故 $h + \frac{1}{h} > \sqrt{k+2}$。

(ii) 設 $P(n)$ 代表命題“$a_n > \sqrt{2n+1}$”。當 $n=1$，$a_1 = 2 > \sqrt{3} = \sqrt{2(1)+1}$，

故 $P(1)$ 成立。假設對某正整數 k，$P(k)$ 成立，即 $a_k > \sqrt{2k+1}$。 則

$$a_{k+1} = a_k + \frac{1}{a_k} > \sqrt{(2k+1)+2} = \sqrt{2(k+1)+1}$$

其中的不等式可從 $a_k > \sqrt{2k+1}$ 及 (i) 推出。故 $P(k+1)$ 也成立。

根據數學歸納法原理，$P(n)$ 對所有正整數 n 都成立。

(iii) 用 (ii) 的結果，有

$$\frac{b_{n+1}}{b_n}=\frac{a_{n+1}/\sqrt{n+1}}{a_n/\sqrt{n}}=\frac{(a_n+1/a_n)\sqrt{n}}{a_n\sqrt{n+1}}=(1+\frac{1}{a_n^2})\frac{\sqrt{n}}{\sqrt{n+1}}$$

$$<(1+\frac{1}{2n+1})\frac{\sqrt{n}}{\sqrt{n+1}}=\frac{2(n+1)\sqrt{n}}{(2n+1)\sqrt{n+1}}=\frac{\sqrt{4n^2+4n}}{\sqrt{4n^2+4n+1}}$$

$$<1$$

5. (a)

$$\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}=abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}=abc\begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

$$=abc(b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}=abc(b-a)(c-a)(c-b)$$

(b) (i) $\begin{cases} x-2y+z=-3 \\ -2x+5y+z=5 \end{cases}\Rightarrow\begin{cases} x-2y+\ z=-3 \\ y+3z=-1 \end{cases}$

設 $z=t$，則方程組的通解可寫成 $x=-7t-5$，$y=-3t-1$，$z=t$，其中 t 是任意實數。

(ii) 把 (i) 的通解代入第三條方程，得到 $(2-7p)t+(1-5p)=q$。

若 $p\neq\frac{2}{7}$，則對任意 q，$t=\frac{5p+q-1}{2-7p}$。故方程組的解為

$x=\frac{-3-7q}{2-7p}$，$y=\frac{1-8p-3q}{2-7p}$，$z=\frac{5p+q-1}{2-7p}$。

若 $p=\frac{2}{7}$，則 $q=(1-5p)=-\frac{3}{7}$。此時，(b)(i) 中方程組的通解就是此方程組的解。

Suggested Answer

1. (a) Since $|BK|^2 = 24$, $|BM|^2 = 21$ and $|MK|^2 = 9$, we have

$$\cos\angle MBK = \frac{|BM|^2 + |BK|^2 - |MK|^2}{2|BM||BK|} = \frac{3}{\sqrt{14}}.$$

(b) (i) Since $BA \perp AEHD$ and $MA \subset AEHD$, we have $BA \perp MA$.

(ii) The volume of $M - ABD = \frac{1}{3}|MN|\left(\frac{1}{2}|AB||AD|\right) = \frac{8}{3}$.

Let h be the distance from point D to the plane AMB. Then

the volume of $D - AMB = \frac{1}{3}h\left(\frac{1}{2}|AM||AB|\right) = \frac{2\sqrt{5}h}{3}$. Hence $h = \frac{4}{\sqrt{5}}$.

2. (a) (i) $f'(x) = 3x^2 - 6x - 9$ and $f''(x) = 6x - 6$

(ii) $f'(x) = 0 \Leftrightarrow 3x^2 - 6x - 9 = 0 \Leftrightarrow 3(x-3)(x+1) = 0 \Leftrightarrow x = 3$ or $x = -1$.

When $x < -1$, $f'(x) > 0$ and hence $f(x)$ is increasing.

When $-1 < x < 3$, $f'(x) < 0$ and hence $f(x)$ is decreasing.

When $3 < x$, $f'(x) > 0$ and hence $f(x)$ is increasing.

Thus, $f(-1) = 7$ is local maximum point and $f(3) = -25$ is a local minimum point.

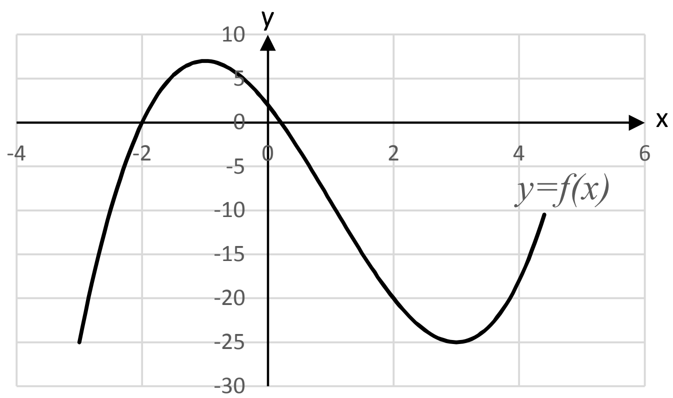
$f''(x) = 0 \Leftrightarrow 6x - 6 = 0 \Leftrightarrow x = 1$.

When $x < 1$, $f''(x) < 0$ and hence $f(x)$ is concave.

When $x > 1$, $f''(x) > 0$ and hence $f(x)$ is convex.

Thus, $f(1) = -9$ is an inflection point.

(iii)



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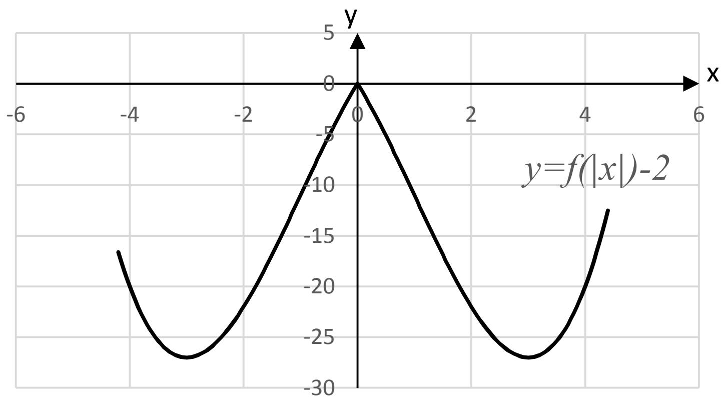

(b) Solving $\begin{cases} y = x^3 - 3x^2 - 9x + 2 \\ y = x + 2 \end{cases}$, we know that the curve $C: y = x^3 - 3x^2 - 9x + 2$

and the line $L: y = x + 2$ intersect at $x = -2$, $x = 0$ and $x = 5$。When $-2 < x < 0$, the curve C is above the line L. When $0 < x < 5$, the curve C is below the line L. Hence the required area is

$$\int_{-2}^{0} (x^3 - 3x^2 - 9x + 2) - (x+2)\ dx + \int_{0}^{5} (x+2) - (x^3 - 3x^2 - 9x + 2)\ dx$$
$$= \int_{-2}^{0} x^3 - 3x^2 - 10x\ dx + \int_{0}^{5} -x^3 + 3x^2 + 10x\ dx$$
$$= \left[\frac{x^4}{4} - x^3 - 5x^2\right]_{-2}^{0} + \left[-\frac{x^4}{4} + x^3 + 5x^2\right]_{0}^{5} \text{。}$$
$$= 8 + \frac{375}{4}$$
$$= \frac{407}{4}$$

3. (a) Suppose the equation of the tangent line L is $y = m(x+1) + 3$. From $\begin{cases} y = (x-1)^2 \\ y = m(x+1) + 3 \end{cases}$, we get $(x-1)^2 = m(x+1) + 3$ and hence

$$x^2 - (2+m)x - (2+m) = 0 \qquad (*)$$

As L is a tangent line of P, (*) has a double root and so its discriminant is 0. Hence we get $[-(2+m)]^2 - 4(1)(-2-m) = 0$, i.e., $m^2 + 8m + 12 = 0$. Thus, $m = -6$ or $m = -2$.

Suppose $m = -6$. From (*) we get $x^2 + 4x + 4 = 0$ and hence $x = -2$. Now, the tangent line L is $y = -6x - 3$. As $x = -2$ we get $y = 9$. Hence the tangent point is $(-2,9)$.

Suppose $m = -2$. From (*) we get $x^2 = 0$ and hence $x = 0$. Now the tangent line L is $y = -2x + 1$. As $x = 0$ we get $y = 1$. Hence the tangent point is $(0,1)$.

(b) Suppose $\tan\theta_1 = -2$ and $\tan\theta_2 = -6$, where $-\frac{\pi}{2} < \theta_2 < \theta_1 < 0$. Then $\alpha = \theta_1 - \theta_2$.

Hence $\tan\alpha = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2} = \frac{(-2)-(-6)}{1+(-2)(-6)} = \frac{4}{13}$.

(c) Suppose $B = (-2,9)$ and $C = (0,1)$. Compute $\tan\alpha = \frac{4}{13} \Rightarrow \sin\alpha = \frac{4}{\sqrt{4^2+13^2}} = \frac{4}{\sqrt{185}}$,

$|AB| = \sqrt{[-2-(-1)]^2 + (9-3)^2} = \sqrt{37}$, $|AC| = \sqrt{[0-(-1)]^2 + (1-3)^2} = \sqrt{5}$.

The area of triangle ABC is $\frac{1}{2}(|AB|\sin\alpha)|AC| = 2$.

4. (a) Suppose $\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$. From $1 = A(x+1) + B(x-1) = (A+B)x + (A-B)$ we get

$\begin{cases} A+B=0 \\ A-B=1 \end{cases}$. Hence $A = \frac{1}{2}$ and $B = -\frac{1}{2}$. Thus,

$$\sum_{n=2}^{10} \frac{1}{n^2-1} = \sum_{n=2}^{10}\left(\frac{1/2}{n-1} - \frac{1/2}{n+1}\right) = \sum_{n=2}^{10}\frac{1/2}{n-1} - \sum_{n=2}^{10}\frac{1/2}{n+1}$$
$$= \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{9}\right) - \frac{1}{2}\left(\frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{11}\right) = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{10} - \frac{1}{11}\right)$$
$$= \frac{36}{55}$$

(b)(i) Since $\left(h + \frac{1}{h}\right)^2 = h^2 + 2 + \frac{1}{h^2} > k + 2 + \frac{1}{h^2} > k + 2$, we have $h + \frac{1}{h} > \sqrt{k+2}$.

(ii) Let $P(n)$ be the proposition "$a_n > \sqrt{2n+1}$". When $n = 1$,

$a_1 = 2 > \sqrt{3} = \sqrt{2(1)+1}$ and thus $P(1)$ is true. Suppose for some integer k,

$P(k)$ is true, i.e., $a_k > \sqrt{2k+1}$. Then

$$a_{k+1} = a_k + \frac{1}{a_k} > \sqrt{(2k+1)+2} = \sqrt{2(k+1)+1},$$

in which the inequality can be deduced from $a_k > \sqrt{2k+1}$ and (i). Hence

$P(k+1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all positive integers *n*.

(iii) Using the result in (ii), we have

$$\frac{b_{n+1}}{b_n} = \frac{a_{n+1}/\sqrt{n+1}}{a_n/\sqrt{n}} = \frac{(a_n + 1/a_n)\sqrt{n}}{a_n\sqrt{n+1}} = (1+\frac{1}{a_n^2})\frac{\sqrt{n}}{\sqrt{n+1}}$$

$$< (1+\frac{1}{2n+1})\frac{\sqrt{n}}{\sqrt{n+1}} = \frac{2(n+1)\sqrt{n}}{(2n+1)\sqrt{n+1}} = \frac{\sqrt{4n^2+4n}}{\sqrt{4n^2+4n+1}}$$

$$< 1$$

5. (a)

$$\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = abc\begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

$$= abc(b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix} = abc(b-a)(c-a)(c-b)$$

(b) (i) $\begin{cases} x-2y+z=-3 \\ -2x+5y+z=5 \end{cases} \Rightarrow \begin{cases} x-2y+\ z=-3 \\ y+3z=-1 \end{cases}$

Let $z=t$. Then the general solution of the system of equations can be written as $x=-7t-5$ ， $y=-3t-1$ ， $z=t$ ， where *t* is any real number.

(ii) Substituting the general solution in (i) into the third equation, we get $(2-7p)t+(1-5p)=q$.

If $p\neq\frac{2}{7}$ then for any q , $t=\frac{5p+q-1}{2-7p}$. Hence the solution of the system of equations is $x=\frac{-3-7q}{2-7p}$, $y=\frac{1-8p-3q}{2-7p}$ ， $z=\frac{5p+q-1}{2-7p}$.

If $p=\frac{2}{7}$ then $q=(1-5p)=-\frac{3}{7}$. Now the general solution of the system of equations in (b)(i) is the solution of this system of equations.