澳門大學
UNIVERSIDADE DE MACAU
UNIVERSITY OF MACAU

澳門理工學院
Instituto Politécnico de Macau
Macao Polytechnic Institute

旅遊學院
INSTITUTO DE FORMAÇÃO TURÍSTICA
Institute for Tourism Studies

澳門科技大學
UNIVERSIDADE DE CIÊNCIA E TECNOLOGIA DE MACAU
MACAU UNIVERSITY OF SCIENCE AND TECHNOLOGY

澳門四高校聯合入學考試（語言科及數學科）

Joint Admission Examination for Macao Four Higher Education Institutions (Languages and Mathematics)

2017/2018 學年試題及參考答案

2017/2018 Examination Paper and Suggested Answer

數學附加卷　Mathematics Supplementary Paper

日期：二〇一七年四月二日　　　　　　　　　　　　考試時間：一小時

注意事項：

1. 考生獲發文件如下:

   1.1 本考卷包括封面共 4 頁

   1.2 答題簿一本

   1.3 草稿紙一張

2. 請於答題簿首頁填寫聯考編號、科目編號及名稱、考場、樓宇、考室及座號。

3. 本考卷共有五條解答題，每題二十分，任擇三題作答。全卷滿分為六十分。

4. 必須在答題簿內作答，寫在其他地方的答案將不會獲評分。

5. 如作答多於三題，只有首三題可得分。

6. 必須將解題步驟清楚寫出。只當答案和所有步驟正確而清楚地表示出來，考生方可獲得滿分。

7. 本考卷的圖形並非按比例繪畫。

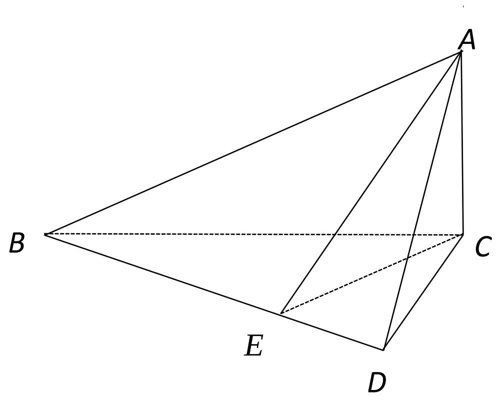
8. 考生可使用不具備輸入計算程式功能之計算機,並於操作計算器時不可發出任何聲響。考生嚴禁使用具列印功能、顯示圖表/文字功能之計算器。

9. 請用藍色或黑色原子筆作答。

10. 考試完畢，考生須交回本考卷、答題簿及草稿紙。

任擇三題作答，每題二十分。

1.



如上圖所示，A-BCD 為一三棱錐，AC 垂直平面 BCD，$\angle BCD = \dfrac{\pi}{2}$，

$|AC| = |CD| = 1$，$|BC| = 2$。設 E 為直線 BD 上的一點，使得 AE 垂直 BD。

(a) (i) 求點 B 至直線 AD 的距離。[提示: 先證明 $|BA| = |BD|$。] (7 分)

(ii) 求點 C 至平面 ABD 的距離。 (5 分)

(b) (i) 證明 CE 垂直 BD。 (2 分)

(ii) 求二面角 A-BD-C，答案以 $\tan^{-1}$ 表示。[提示: 求 $|EC|$。] (6 分)

2. (a) 一體積為 $72\ \text{cm}^3$ 的長方體，其長、闊、高分別為 $2x$ cm、x cm 及 h cm，其中 $1 \le x \le 6$。

(i) 把長方體的表面面積表示成 x 的函數 $S(x)$。 (3 分)

(ii) 求 $\dfrac{dS}{dx}$ 及 $\dfrac{d^2S}{dx^2}$，$1 < x < 6$。 (2 分)

(iii) 繪出曲線 $y = S(x)$。圖中標示出局部極大點、局部極小點和拐點。 (6 分)

(iv) 求 $S(x)$ 的最大及最小值。 (2 分)

(b) 求 a 的值，使得由曲線 $y = ax(a - x)$ 和 x-軸所包圍的區域的面積為 $\dfrac{8}{3}$。 (7 分)

3. 設直線 $L: y = mx + c$ ($m \neq 0$) 為拋物線 $P: y^2 = 2x$ 的切線。

(a) 證明 $c = \dfrac{1}{2m}$。 (6 分)

(b) 設 L_1 及 L_2 為拋物線 P 的兩條不同的切線，其斜率分別為非零數 m_1 及 m_2。

(i) 求 L_1 及 L_2 的交點 A，答案以 m_1 及 m_2 表示。 (7 分)

(ii) 設 $m_1 + m_2 = 1$，求點 A 的軌跡。 (7 分)

4. 設 $i = \sqrt{-1}$。

(a) 設 $z = \cos\theta + i\sin\theta$，其中 θ 為實數。

(i) 用數學歸納法，證明對任意正整數 n，$z^n = \cos n\theta + i\sin n\theta$。 (6 分)

(ii) 證明對任意正整數 n，$z^n + \dfrac{1}{z^n} = 2\cos n\theta$。 (5 分)

(b) 用 (a)(ii) 的結果，證明恆等式 $8\cos^4\theta = \cos 4\theta + 4\cos 2\theta + 3$。 (5 分)

(c) 用 (b) 的結果，求方程 $\cos 4\theta + 4\cos 2\theta + 1 = 0$ 的通解。 (4 分)

5. (a) 因式分解行列式 $\begin{vmatrix} 1 & a^2 & b+c \\ 1 & b^2 & c+a \\ 1 & c^2 & a+b \end{vmatrix}$。 (8 分)

(b) 求方程組 $\begin{cases} x - 2y + 5z = 2 \\ 3x + y + z = 13 \end{cases}$ 的通解。 (3 分)

(c) 已知以 x、y 和 z 為未知量的方程組:

$$(E): \begin{cases} x - 2y + 5z = 2 \\ 3x + y + z = 13 \\ px + y - z = 3 \end{cases},$$

其中 p 為常數。

(i) 求 p 的取值範圍，使得 (E) 有唯一解。 (3 分)

(ii) 用 (b) 的通解，求 p 的取值範圍，使得 (E) 有唯一正數解 (x, y, z)，即 $x > 0, y > 0, z > 0$。 (6 分)

全卷完

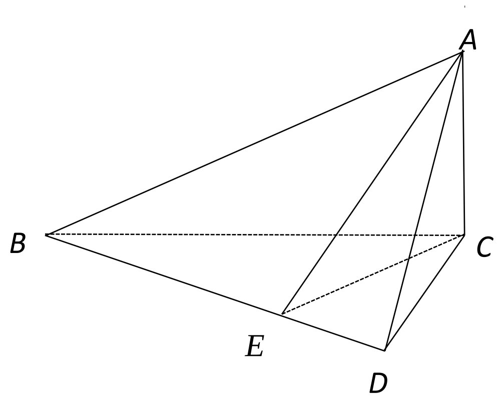
Date: April 2, 2017 Examination time: one hour

Instructions:

1. Each candidate is provided with the following documents:
   - 1.1 Question paper including cover page – 4 pages
   - 1.2 One answer booklet
   - 1.3 One sheet of draft paper
2. Fill in your JAE No., subject code and title, campus, building, room and seat no. on the front page of the answer booklet.
3. There are 5 questions in this paper, each carries 20 marks. Answer any 3 questions. Full mark of this paper is 60.
4. Put your answers in the answer booklet provided. Answers put elsewhere will not be marked.
5. If more than 3 questions are answered, only the first 3 will be marked.
6. Show all your steps in getting to the answer. Full credits will be given only if the answer and all the steps are correct and clearly shown.
7. The diagrams in this examination paper are not drawn to scale.
8. Candidates may use a non-programmable calculator which is silent during operation. Candidates are strictly prohibited from using a calculator with print-out, graphic/word-display functions.
9. Answer the questions with a blue or black ball pen.
10. Candidates must return the question paper, answer booklet and draft paper at the end of the examination.

Answer any 3 questions, each carries 20 marks.

1.



In the above figure, *A*-*BCD* is a triangular pyramid, *AC* is perpendicular to plane *BCD*, $\angle BCD = \frac{\pi}{2}$, $|AC| = |CD| = 1$ and $|BC| = 2$. Let *E* be a point on *BD* such that *AE* is perpendicular *BD*.

(a) (i) Find the distance from point *B* to line *AD*.

[Hint: Show firstly that $|BA| = |BD|$.] (7 marks)

(ii) Find the distance from point *C* to plane ABD. (5 marks)

(b) (i) Prove that *CE* is perpendicular to *BD*. (2 marks)

(ii) Find the dihedral angle *A*-*BD*-*C*. Express your answer in terms of $\tan^{-1}$.

[Hint: Find $|EC|$.] (6 marks)

2. (a) A rectangular block has volume 72 cm^3. Its length, width and height are 2*x* cm, *x* cm and *h* cm, respectively, where $1 \le x \le 6$.

(i) Express the surface area of the block as a function $S(x)$ in *x*. (3 marks)

(ii) Find $\frac{dS}{dx}$ and $\frac{d^2S}{dx^2}$, $1 < x < 6$. (2 marks)

(iii) Sketch the curve $y = S(x)$. Indicate in the graph the local maximum points, local minimum points and inflection points. (6 marks)

(iv) Find the maximum and minimum values of $S(x)$. (2 marks)

(b) Find the value(s) of *a* such that the area of the region bounded the curve $y = ax(a - x)$ and the *x*-axis is $\frac{8}{3}$. (7 marks)

3. Let $L: y = mx + c$ $(m \neq 0)$ be a tangent line of the parabola $P: y^2 = 2x$.

(a) Show that $c = \dfrac{1}{2m}$. (6 marks)

(b) Suppose L_1 and L_2 are two different tangent lines of the parabola P and their slopes are nonzero numbers m_1 and m_2, respectively.

(i) Find the intersection point *A* of L_1 and L_2. Express your answer in terms of m_1 and m_2. (7 marks)

(ii) Suppose $m_1 + m_2 = 1$. Find the locus of point *A*. (7 marks)

4. Let $i = \sqrt{-1}$.

(a) Let $z = \cos\theta + i\sin\theta$, where θ is a real number.

(i) Use mathematical induction to show that $z^n = \cos n\theta + i\sin n\theta$ for all positive integers *n*. (6 marks)

(ii) Show that for all positive integers *n*, $z^n + \dfrac{1}{z^n} = 2\cos n\theta$. (5 marks)

(b) Using the result in (a) (ii), prove the identity $8\cos^4\theta = \cos 4\theta + 4\cos 2\theta + 3$. (5 marks)

(c) Using the result in (b), find the general solution of $\cos 4\theta + 4\cos 2\theta + 1 = 0$. (4 marks)

5. (a) Factorize the determinant $\begin{vmatrix} 1 & a^2 & b+c \\ 1 & b^2 & c+a \\ 1 & c^2 & a+b \end{vmatrix}$. (8 marks)

(b) Find the general solution of the system of equations $\begin{cases} x - 2y + 5z = 2 \\ 3x + y + z = 13 \end{cases}$. (3 marks)

(c) Given a system of equations with unknowns *x*, *y* and *z*:

$$(E): \begin{cases} x - 2y + 5z = 2 \\ 3x + y + z = 13, \\ px + y - z = 3 \end{cases}$$

where *p* is a constant.

(i) Find the range of *p* such that (*E*) has a unique solution. (3 marks)

(ii) Using the general solution in (b), find the range of *p* such that (*E*) has a unique positive solution (x, y, z), i.e. $x > 0, y > 0, z > 0$. (6 marks)

End of Paper

參考答案：

1. (a) (i) 因 $|AB|=\sqrt{|AC|^2+|BC|^2}=\sqrt{5}$ 及 $|BD|=\sqrt{|BC|^2+|CD|^2}=\sqrt{5}$，故 ΔBAD 是等腰三角形，其中 $|AB|=|BD|$。設 F 為 AD 的中點，則 $BF\perp AD$ 及 $|AF|=\dfrac{|AD|}{2}=\dfrac{\sqrt{2}}{2}$。因此，點 B 至直線 AD 的距離為

$$|BF|=\sqrt{|AB|^2-|AF|^2}=\frac{3\sqrt{2}}{2}。$$

(ii) 設 h 為點 C 至平面 ABD 的距離，則有

$$A-BCI \text{ 的體積 } =\frac{1}{3}|AC|\left(\frac{1}{2}|BC||CD|\right)=\frac{1}{3} \text{ 及}$$

$$C-ABD \text{ 的體積}=\frac{1}{3}h\left(\frac{1}{2}|AD||BF|\right)=\frac{h}{2}，\text{故 } h=\frac{2}{3}。$$

(b) (i) 因 $AC\perp BCD$，故 $AC\perp BD$。連同 $AE\perp BD$，得知 $BD\perp ACE$。由此得出 $CE\perp BD$。

(ii) 從 (i)，得知二面角 $A\text{-}BD\text{-}C$ 與 $\angle AEC$ 相等。由

$$\frac{1}{2}\sqrt{5}|CE|=\frac{1}{2}|BD||CE|=\Delta BCI \text{ 的面積 } =\frac{1}{2}|BC||CD|=1，$$

得出 $|CE|=\dfrac{2\sqrt{5}}{5}$，故二面角 $A\text{-}BD\text{-}C$ $=\angle AEC=\tan^{-1}\dfrac{|AC|}{|CE|}=\tan^{-1}\dfrac{\sqrt{5}}{2}$。

2. (a) (i) 長方體的表面面積為 $2[(2x)x+(2x)h+xh]=4x^2+6xh\text{ cm}^2$。由體積的條件得到 $72=(2x)(x)(h)$，即 $h=\dfrac{36}{x^2}$，故 $S(x)=4x^2+\dfrac{216}{x}$。

(ii) $\dfrac{dS}{dx}=8x-\dfrac{216}{x^2}$ 及 $\dfrac{d^2S}{dx^2}=8+\dfrac{432}{x^3}$

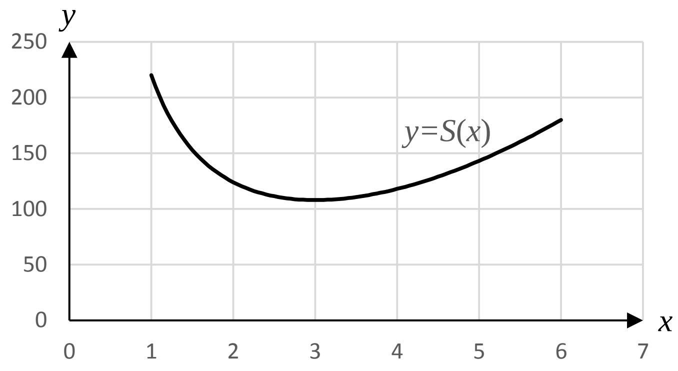
(iii) $\dfrac{dS}{dx}=0\Leftrightarrow 8x-\dfrac{216}{x^2}=0\Leftrightarrow x^3-27=0\Leftrightarrow x=3$。

當 $1<x<3$ 時，$\dfrac{dS}{dx}<0$， 故 $S(x)$ 是遞減的。

當 $3<x<6$ 時，$\dfrac{dS}{dx}>0$， 故 $S(x)$ 是遞增的。

因此，$S(3)=108$ 是一局部極小點。

$\dfrac{d^2S}{dx^2}=8+\dfrac{432}{x^3}>0,\ 1<x<6,\quad\Rightarrow$ 此曲線沒有拐點，且是凸的。



(iv) $S(x)$ 的最大值是 $S(1)=220$，其最小值是 $S(3)=108$。

(b) 解 $ax(a-x)=0$，得知曲線 $y=ax(a-x)$ 交 x-軸於 $x=0$ 和 $x=a$。

用面積的條件，得出

$$\left|\int_0^a a^2x-ax^2dx\right|=\frac{8}{3}\Leftrightarrow\left|\left[\frac{a^2x^2}{2}-\frac{ax^3}{3}\right]_0^a\right|=\frac{8}{3}\Leftrightarrow\left|\frac{a^4}{6}\right|=\frac{8}{3}\Leftrightarrow a=\pm 2\text{ 。}$$

3. (a) 解 $\begin{cases} y=mx+c \\ y^2=2x \end{cases}$，由 $(mx+c)^2=2x$ 得

$$m^2x^2+(2mc-2)x+c^2=0\text{ 。}\qquad (*)$$

因 L 是 P 的切線，故 (*) 有二重根，其判別式等於 0。因此得

$(2mc-2)^2-4m^2c^2=0$，即 $c=\dfrac{1}{2m}$。

(b) (i) 設 L_1 及 L_2 分別為

$$y=m_1x+\frac{1}{2m_1}\quad\text{及}\quad y=m_2x+\frac{1}{2m_2}\text{ ，}$$

它們的交點 A 是 $\left(\dfrac{1}{2m_1m_2},\dfrac{m_1+m_2}{2m_1m_2}\right)$。

(ii) 因 $m_1+m_2=1$，故點 A 是 $\left(\dfrac{1}{2m_1m_2},\dfrac{1}{2m_1m_2}\right)$，且有

$$m_1m_2=m_1(1-m_1)=m_1-m_1^2=\frac{1}{4}-\left(m_1-\frac{1}{2}\right)^2\le\frac{1}{4}\text{ 。}$$

由於 $m_1\ne m_2$，得知 $m_1\ne\dfrac{1}{2}$，故 $m_1m_2<\dfrac{1}{4}$。所以，

$$\frac{1}{2m_1m_2}>2\ (\text{當 } 0<m_1m_2<\frac{1}{4})\quad\text{或}\quad\frac{1}{2m_1m_2}<0\ (\text{當 } m_1m_2<0)\text{。}$$

因此，點 A 的軌跡是 $\{(t,t):t>2 \text{ 或 } t<0\}$。

4. (a) (i) 設 $S(n)$ 代表命題“$z^n=\cos n\theta+i\sin n\theta$”。當 $n=1$，$S(1)$ 明顯成立。假設對某正整數 k，$S(k)$ 成立，即 $z^k=\cos k\theta+i\sin k\theta$。則

$$\begin{aligned}
z^{k+1}&=z^k z\\
&=(\cos k\theta+i\sin k\theta)(\cos\theta+i\sin\theta)\\
&=(\cos k\theta\cos\theta-\sin k\theta\sin\theta)+i(\cos k\theta\sin\theta+\sin k\theta\cos\theta)\\
&=\cos[(k+1)\theta]+i\sin[(k+1)\theta]
\end{aligned}$$ 。

故 $S(k+1)$ 也成立。根據數學歸納法原理，$S(n)$ 對所有正整數 n 都成立。

(ii) 因

$$z^n\overline{z^n}=(\cos n\theta+i\sin n\theta)(\cos n\theta-i\sin n\theta)=\cos^2 n\theta+\sin^2 n\theta=1，$$

故 $\dfrac{1}{z^n}=\overline{z^n}=(\cos n\theta-i\sin n\theta)$，從而得到 $z^n+\dfrac{1}{z^n}=2\cos n\theta$。

(b) 用 (a)(ii) 的結果，

$$\begin{aligned}
\cos^4\theta&=\left[\frac{1}{2}\left(z+\frac{1}{z}\right)\right]^4\\
&=\frac{1}{16}\left(z^4+4z^2+6+4\frac{1}{z^2}+\frac{1}{z^4}\right)\\
&=\frac{1}{16}\left[\left(z^4+\frac{1}{z^4}\right)+4\left(z^2+\frac{1}{z^2}\right)+6\right]\\
&=\frac{1}{16}(2\cos 4\theta+8\cos 2\theta+6)
\end{aligned}$$ 。

由此可知恆等式成立。

(c) 用 (b) 的結果，

$$\begin{aligned}
&\cos 4\theta+4\cos 2\theta+1=0\\
\Leftrightarrow\ &\cos 4\theta+4\cos 2\theta+3=2\\
\Leftrightarrow\ &8\cos^4\theta=2\\
\Leftrightarrow\ &\cos\theta=\pm\frac{\sqrt{2}}{2}\\
\Leftrightarrow\ &\theta=k\pi\pm\frac{\pi}{4}
\end{aligned}$$

，其中 k 為整數。

5. (a)

$$\begin{vmatrix} 1 & a^2 & b+c \\ 1 & b^2 & a+c \\ 1 & c^2 & a+b \end{vmatrix} = \begin{vmatrix} 1 & a^2 & b+c \\ 0 & b^2-a^2 & a-b \\ 1 & c^2 & a+b \end{vmatrix} = (a-b)\begin{vmatrix} 1 & a^2 & b+c \\ 0 & -a-b & 1 \\ 1 & c^2 & a+b \end{vmatrix}$$

$$= (a-b)\begin{vmatrix} 1 & a^2 & b+c \\ 0 & -a-b & 1 \\ 0 & c^2-a^2 & a-c \end{vmatrix} = (a-b)(a-c)\begin{vmatrix} 1 & a^2 & b+c \\ 0 & -a-b & 1 \\ 0 & -c-a & 1 \end{vmatrix}$$

$$= (a-b)(a-c)(c-b)$$

(b)

$$\begin{cases} x-2y+5z=2 \\ 3x+\ y+\ z=13 \end{cases} \Rightarrow \begin{cases} x-2y+5z=2 \\ \qquad y-2z=1 \end{cases}$$

設 $z=t$，則方程組的通解可寫成 $x=4-t$，$y=1+2t$，$z=t$，其中 t 是任意實數。

(c) (i) 方程組 (E) 有唯一解當且僅當 $\begin{vmatrix} 1 & -2 & 5 \\ 3 & 1 & 1 \\ p & 1 & -1 \end{vmatrix} \neq 0$，即 $p \neq 1$。

(ii) 因 (E) 的首兩條方程有正數解，用 (b) 的通解，得知 $x=4-t>0$，$y=1+2t>0$，$z=t>0$，從而亦知 $0<t<4$。把通解代入第三條方程，得 $p(4-t)+(1+2t)-t=3$，即 $p=1-\dfrac{2}{4-t}$。由 $0<t<4$，得出 $p<\dfrac{1}{2}$。從 (i) 得知此正數解是唯一的。

Suggested Answer

1. (a) (i) Since $|AB| = \sqrt{|AC|^2 + |BC|^2} = \sqrt{5}$ and $|BD| = \sqrt{|BC|^2 + |CD|^2} = \sqrt{5}$, ΔBAD is an isosceles triangle with $|AB| = |BD|$. Let F be the mid-point of AD. Then, $BF \perp AD$ and $|AF| = \frac{|AD|}{2} = \frac{\sqrt{2}}{2}$. Hence, the distance from point B to line AD is $|BF| = \sqrt{|AB|^2 - |AF|^2} = \frac{3\sqrt{2}}{2}$.

(ii) Let h be the distance from point C to plane ABD. Then,

volume of $A-BCD$ $= \frac{1}{3}|AC|\left(\frac{1}{2}|BC||CD|\right) = \frac{1}{3}$ and

volume of $C-ABD$ $= \frac{1}{3}h\left(\frac{1}{2}|AD||BF|\right) = \frac{h}{2}$. Hence $h = \frac{2}{3}$.

(b) (i) Since $AC \perp BCD$, we know that $AC \perp BD$. Together with $AE \perp BD$, we have $BD \perp ACE$. Hence, $CE \perp BD$.

(ii) From (i), we know that the dihedral angle A-BD-C and $\angle AEC$ are equal. From

$$\frac{1}{2}\sqrt{5}|CE| = \frac{1}{2}|BD||CE| = \text{area of } \Delta BCD = \frac{1}{2}|BC||CD| = 1,$$

$|CE| = \frac{2\sqrt{5}}{5}$. Hence, the dihedral angle A-BD-C $= \angle AEC = \tan^{-1}\frac{|AC|}{|CE|} = \tan^{-1}\frac{\sqrt{5}}{2}$.

2. (a) (i) The surface area of the rectangular block is $2[(2x)x + (2x)h + xh] = 4x^2 + 6xh$ cm^2.

Form the condition on the volume, we get $72 = (2x)(x)(h)$, i.e., $h = \frac{36}{x^2}$. Hence,

$S(x) = 4x^2 + \frac{216}{x}$.

(ii) $\frac{dS}{dx} = 8x - \frac{216}{x^2}$ and $\frac{d^2S}{dx^2} = 8 + \frac{432}{x^3}$

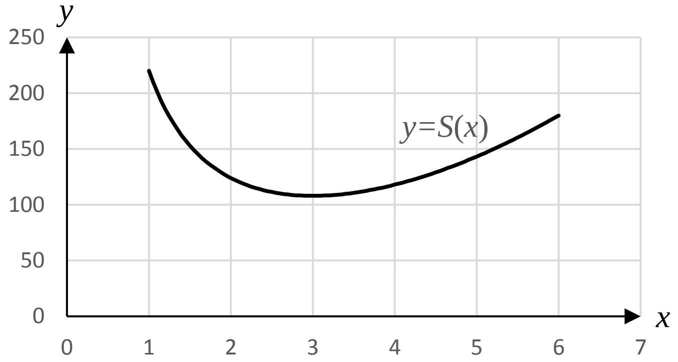
(iii) $\frac{dS}{dx} = 0 \Leftrightarrow 8x - \frac{216}{x^2} = 0 \Leftrightarrow x^3 - 27 = 0 \Leftrightarrow x = 3$.

When $1 < x < 3$, $\frac{dS}{dx} < 0$. So $S(x)$ is decreasing.

When $3 < x < 6$, $\frac{dS}{dx} > 0$. So $S(x)$ is increasing.

Hence $S(3) = 108$ is a local minimum point.

$\frac{d^2S}{dx^2} = 8 + \frac{432}{x^3} > 0,\ 1 < x < 6,$ $\Rightarrow$ The curve has no inflection point and is convex.



(iv) The maximum value of $S(x)$ is $S(1)=220$, and its minimum value is $S(3)=108$.

(b) Solving $ax(a-x)=0$, we know that the curve $y=ax(a-x)$ intersects the x-axis at $x=0$ and $x=a$. Using the condition on the area, we get

$$\left|\int_0^a a^2x - ax^2 dx\right| = \frac{8}{3} \Leftrightarrow \left|\left[\frac{a^2x^2}{2} - \frac{ax^3}{3}\right]_0^a\right| = \frac{8}{3} \Leftrightarrow \left|\frac{a^4}{6}\right| = \frac{8}{3} \Leftrightarrow a = \pm 2 .$$

3. (a) We solve $\begin{cases} y = mx + c \\ y^2 = 2x \end{cases}$. From $(mx+c)^2 = 2x$ we get

$$m^2x^2 + (2mc-2)x + c^2 = 0 . \qquad (*)$$

Since L is a tangent line of P, (*) has a double root and so its discriminant is 0.

Hence we get $(2mc-2)^2 - 4m^2c^2 = 0$, i.e. $c = \dfrac{1}{2m}$.

(b) (i) Let L_1 and L_2 be, respectively,

$$y = m_1x + \frac{1}{2m_1} \quad \text{and} \quad y = m_2x + \frac{1}{2m_2} .$$

Their intersection point A is $\left(\dfrac{1}{2m_1m_2}, \dfrac{m_1+m_2}{2m_1m_2}\right)$.

(ii) Since $m_1+m_2=1$, the point A is $\left(\dfrac{1}{2m_1m_2}, \dfrac{1}{2m_1m_2}\right)$. Moreover, we have

$$m_1m_2 = m_1(1-m_1) = m_1 - m_1^2 = \frac{1}{4} - \left(m_1 - \frac{1}{2}\right)^2 \le \frac{1}{4} .$$

As $m_1 \ne m_2$, we know that $m_1 \ne \dfrac{1}{2}$ and hence $m_1m_2 < \dfrac{1}{4}$. Thus,

$$\frac{1}{2m_1m_2} > 2 \ \ (\text{when } 0 < m_1m_2 < \frac{1}{4}) \quad \text{or} \quad \frac{1}{2m_1m_2} < 0 \ \ (\text{when } m_1m_2 < 0).$$

Therefore, the locus of point A is $\{(t,t) : t > 2 \text{ or } t < 0\}$.

4. (a) (i) Let $S(n)$ denote the proposition "$z^n = \cos n\theta + i\sin n\theta$". When $n=1$, $S(1)$ is obviously true. Suppose $S(k)$ is true for some positive integer k, i.e. $z^k = \cos k\theta + i\sin k\theta$. Then,

$$\begin{aligned} z^{k+1} &= z^k z \\ &= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta) \\ &= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\cos k\theta\sin\theta + \sin k\theta\cos\theta) \\ &= \cos[(k+1)\theta] + i\sin[(k+1)\theta]. \end{aligned}$$

Thus, $S(k+1)$ is true. By the principle of mathematical induction, $S(n)$ is true for all positive integers *n*.

(ii) Since

$$z^n\overline{z^n} = (\cos n\theta + i\sin n\theta)(\cos n\theta - i\sin n\theta) = \cos^2 n\theta + \sin^2 n\theta = 1,$$

we get $\dfrac{1}{z^n} = \overline{z^n} = (\cos n\theta - i\sin n\theta)$, from which we obtain $z^n + \dfrac{1}{z^n} = 2\cos n\theta$.

(b) Using the result in (a)(ii),

$$\begin{aligned} \cos^4\theta &= \left[\frac{1}{2}\left(z+\frac{1}{z}\right)\right]^4 \\ &= \frac{1}{16}\left(z^4 + 4z^2 + 6 + 4\frac{1}{z^2} + \frac{1}{z^4}\right) \\ &= \frac{1}{16}\left[\left(z^4+\frac{1}{z^4}\right) + 4\left(z^2+\frac{1}{z^2}\right) + 6\right] \\ &= \frac{1}{16}(2\cos 4\theta + 8\cos 2\theta + 6). \end{aligned}$$

Hence we know that the identity is true.

(c) Using the result in (b),

$$\begin{aligned} &\cos 4\theta + 4\cos 2\theta + 1 = 0 \\ \Leftrightarrow\ &\cos 4\theta + 4\cos 2\theta + 3 = 2 \\ \Leftrightarrow\ &8\cos^4\theta = 2 \\ \Leftrightarrow\ &\cos\theta = \pm\frac{\sqrt{2}}{2} \\ \Leftrightarrow\ &\theta = k\pi \pm \frac{\pi}{4}, \quad \text{where } k \text{ is an integer.} \end{aligned}$$

5. (a)

$$\begin{vmatrix} 1 & a^2 & b+c \\ 1 & b^2 & a+c \\ 1 & c^2 & a+b \end{vmatrix} = \begin{vmatrix} 1 & a^2 & b+c \\ 0 & b^2-a^2 & a-b \\ 1 & c^2 & a+b \end{vmatrix} = (a-b)\begin{vmatrix} 1 & a^2 & b+c \\ 0 & -a-b & 1 \\ 1 & c^2 & a+b \end{vmatrix}$$

$$= (a-b)\begin{vmatrix} 1 & a^2 & b+c \\ 0 & -a-b & 1 \\ 0 & c^2-a^2 & a-c \end{vmatrix} = (a-b)(a-c)\begin{vmatrix} 1 & a^2 & b+c \\ 0 & -a-b & 1 \\ 0 & -c-a & 1 \end{vmatrix}$$

$$= (a-b)(a-c)(c-b)$$

(b)

$$\begin{cases} x-2y+5z=2 \\ 3x+\ \ y+\ \ z=13 \end{cases} \Rightarrow \begin{cases} x-2y+5z=2 \\ \qquad y-2z=1 \end{cases}$$

Letting $z=t$, the solution of the system of equations can be expressed as $x=4-t$, $y=1+2t$, $z=t$, where t is any real number.

(c) (i) The system (E) has a unique solution if and only if $\begin{vmatrix} 1 & -2 & 5 \\ 3 & 1 & 1 \\ p & 1 & -1 \end{vmatrix} \neq 0$, i.e. $p \neq 1$.

(ii) For the first two equations of (E) to have a positive solution, using the general solution obtained in (b), we have $x=4-t>0$, $y=1+2t>0$, $z=t>0$, from which we deduce that $0<t<4$. Substituting the general solution to the third equation, we get $p(4-t)+(1+2t)-t=3$ and hence $p=1-\dfrac{2}{4-t}$. From $0<t<4$, we get $p<\dfrac{1}{2}$. From (i), we know that this positive solution is unique.