2015/2016 學年 入學考試試題

第一部分: 七題全部作答。

1. (a) 求在 $(1+x)^4(1+x^2)^4$ 的展開式中 x^5 的係數。 (2 分)

   (b) 設 $n \geq 2$ 為整數。在 $(1+x)^n$ 的展開式中，x^4 的係數是 x^2 的係數的 6 倍， 求 n 的值。 (4 分)

2. 設函數 $f(x)=\dfrac{1}{\sqrt{\ln(-x^2+8x-6)}}$ 。

   (a) 求 $f(x)$ 的定義域。 (4 分)
   (b) 求 $f(x)$ 的值域。 (4 分)

3. (a) 用數學歸納法，證明對任意正整數 n，有

   $$1^3+2^3+\cdots+n^3=\frac{1}{4}n^2(n+1)^2 \text{ 。}$$ (4 分)

   (b) 用 (a) 的結果，對任意正整數 n，求以下和的公式：

   $$1^3-2^3+3^3-4^3+\cdots+(-1)^{r+1}r^3+\cdots-(2n)^3 \text{ 。}$$ (4 分)

4. 設 A 和 B 為實數，且 $A+B=\dfrac{2\pi}{3}$ 及 $\cos A+\cos B=-2\sqrt{2}\cos A\cos B$ 。

   (a) 證明 $\cos\dfrac{A-B}{2}=\dfrac{\sqrt{2}}{2}-\sqrt{2}\cos(A-B)$ 。 (3 分)

   (b) 求 $\cos\dfrac{A-B}{2}$ 的值。 (4 分)

5. 解不等式 $\dfrac{4^x-5\cdot 2^x+2}{2^x-3}>4$ 。 (8 分)

6. 設 $\{a_n\}_{n=1}^{\infty}$ 為一等差數列，其公差為負數。設 $a_1=9$ 及 a_1、a_2+1、$4a_3$ 為一等比數列的連續三項。

(a) 求此等差數列的通項 a_n。 (4 分)

(b) 對任意正整數 n，求 $|a_1|+|a_2|+\cdots+|a_n|$。 (4 分)

7. 盒子中有 9 張卡片，紅色、藍色、白色各 3 張，每種顏色的卡片分別標號為 1、2、3。

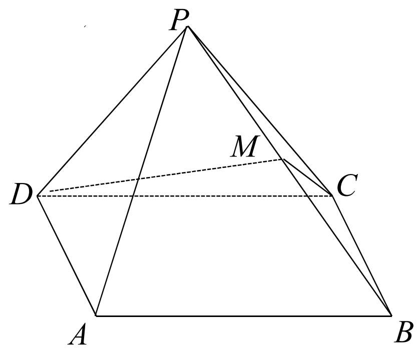
(a) 從盒中隨機抽出卡片 3 張，求這 3 張卡片的標號之和小於 5 的概率。(3 分)

(b) 從盒中隨機抽出卡片 2 張，求這 2 張卡片顏色不同且標號之和小於 4 的概率。 (4 分)

第二部分:任擇三題作答，每題十六分。

8.

圖 I



如圖 I，在四棱錐 $P\text{-}ABCD$ 中，底 $ABCD$ 是菱形，且 $\angle ADC = \frac{\pi}{3}$。$PDC$

是邊長

為 2 的等邊三角形，且與 $ABCD$ 垂直。設 M 為 PB 的中點。

(a) 求 PA 與 $ABCD$ 所成角的大小。 (6 分)

(b) (i) 設 N 為 PA 的中點及 E 為 CD 的中點。證明 $MNEC$ 為一長方形，且與 PA 垂直。 (6 分)

(ii) 求三棱錐 $P\text{-}DMC$ 的體積。 (4 分)

9. (a) 一正圓柱的底半徑和高之和為 36 cm。設此正圓柱的底半徑為 x cm 及體積為 $V(x)$ cm^3，其中 $V(x)$ 為 x 的函數。

(i) 求 $V(x)$。 (2 分)

(ii) 求 $V'(x)$ 及 $V''(x)$。 (2 分)

(iii) 繪出曲線 $y=V(x)$。圖中給出 $V(x)$ 的局部極大點、局部極小點和拐點。

(7 分)

(b) 求由曲線 $y=x^2+x$ 和直線 $y=3-x$ 所包圍的區域的面積。 (5 分)

10. 已知圓 C: $x^2+y^2-2x+4y-4=0$ 。設直線 $L: y=x+b$ 與 C 交於不同的兩點 $A(x_1,y_1)$ 及 $B(x_2,y_2)$。

(a) 求 C 的圓心及半徑。 (2 分)

(b) (i) 證明 x_1 和 x_2 滿足方程 $2x^2+(2b+2)x+(b^2+4b-4)=0$。 (1 分)

(ii) 以 b 表 x_1+x_2、x_1x_2、y_1+y_2 和 y_1y_2。 (4 分)

(c) 求 A 與 B 的距離，答案以 b 表示。 (5 分)

(d) 若以線段 AB 為直徑的圓通過原點 O，求 b 的值。 (4 分)

11. 設 $i=\sqrt{-1}$ 。

(a) 設複數 z 滿足 $4z+2\bar{z}=3\sqrt{3}+i$。

(i) 求 z，並按極式 $r(\cos\alpha+i\sin\alpha)$ 表示，其中 $-\pi<\alpha\le\pi$ 。 (5 分)

(ii) 設 $u=\sin\theta+i\cos\theta$ ，$-\pi<\theta\le\pi$ 。求 $|z-u|$ 的取值範圍。(4 分)

(b) 設複數 w 滿足 $w^7=1$，$w\neq 1$ 及 w 的輻角為 β。

(i) 證明 $1+w+w^2+w^3+w^4+w^5+w^6=0$。 (1 分)

(ii) 證明對任意正整數 n，$w^n+w^{-n}=2\cos(n\beta)$。 (3 分)

(iii) 求 $\cos\beta+\cos(2\beta)+\cos(3\beta)$ 的值。 (3 分)

12. (a) 因式分解行列式 $\begin{vmatrix} a^2 & b^2 & c^2 \\ (1+a)^2 & (1+b)^2 & (1+c)^2 \\ (2+a)^2 & (2+b)^2 & (2+c)^2 \end{vmatrix}$。 (6 分)

(b) 給出以 x、y 和 z 為未知量的方程組 (E):

$$(E)\begin{cases} x + py + pz = 1 \\ px + y + pz = q \\ px + py + z = 1 \end{cases}。$$

(i) 求 p 的值使得 (E) 有唯一解。 (4 分)

(ii) 對使得 (E) 有多於一個解的 p 及 q 的值，求 (E) 的通解。 (6 分)

全卷完

2015/2016 ADMISSION EXAMINATION PAPER

Section I. Answer all 7 questions.

1. (a) In the expansion of $(1+x)^4(1+x^2)^4$, find the coefficient of x^5. (2 marks)

   (b) Let n $\geq$ 2 be an integer. In the expansion of $(1+x)^n$, the coefficient of x^4 is 6 times of the coefficient of x^2. Find the value of n. (4 marks)

2. Let $f(x)=\dfrac{1}{\sqrt{\ln(-x^2+8x-6)}}$ be a function.

   (a) Find the domain of $f(x)$. (4 marks)
   (b) Find the range of $f(x)$. (4 marks)

3. (a) Use mathematical induction to show that for any positive integer n,

$$1^3+2^3+\cdots+n^3=\frac{1}{4}n^2(n+1)^2 .$$ (4 marks)

   (b) Using the result of (a), for any positive integer n, find a formula for the following sum:

$$1^3-2^3+3^3-4^3+\cdots+(-1)^{r+1}r^3+\cdots-(2n)^3.$$ (4 marks)

4. Let A and B be real numbers such that $A+B=\dfrac{2\pi}{3}$ and $\cos A+\cos B=-2\sqrt{2}\cos A\cos B$.

   (a) Show that $\cos\dfrac{A-B}{2}=\dfrac{\sqrt{2}}{2}-\sqrt{2}\cos(A-B)$. (3 marks)

   (b) Find the value of $\cos\dfrac{A-B}{2}$. (4 marks)

5. Solve the inequality $\dfrac{4^x-5\cdot 2^x+2}{2^x-3}>4$. (8 marks)

6. Let $\{a_n\}_{n=1}^{\infty}$ be an arithmetic progression with negative common difference. Suppose $a_1 = 9$ and $a_1, a_2 + 1, 4a_3$ are in geometric progression.

(a) Find the general term a_n of this arithmetic progression. (4 marks)

(b) For any positive integer n, find $|a_1| + |a_2| + \cdots + |a_n|$. (4 marks)

7. A box contains 9 cards. Among them, 3 cards are red, 3 cards are blue and 3 cards are white. The cards of each colour are labeled 1, 2, 3.

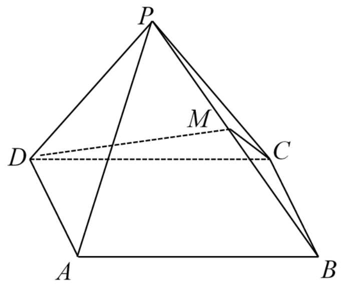
(a) Three cards are drawn randomly from the box. Find the probability that the sum of the numbers on these three cards is less than 5. (3 marks)

(b) Two cards are drawn randomly from the box. Find the probability that the two cardshave different colours and the sum of the numbers on these two cards is less than 4. (4 marks)

Section II Answer any three questions. Each question carries 16 marks.

8.

Figure I



In Figure I, *P-ABCD* is a pyramid. Base *ABCD* is a rhombus with $\angle ADC = \frac{\pi}{3}$. *PDC* is an equilateral triangle with side length 2 and is perpendicular to *ABCD*. Let *M* be the midpoint of *PB*.

(a) Find the angle between *PA* and *ABCD*. (6 marks)

(b) (i) Let *N* be the midpoint of *PA* and *E* be the midpoint of *CD*. Show that *MNEC* is a rectangle and is perpendicular to *PA*. (6 marks)

(ii) Find the volume of the triangular pyramid *P-DMC*. (4 marks)

9. (a) The sum of the base radius and the height of a circular cylinder is 36 cm. Suppose the base radius is *x* cm and the volume is $V(x)$ cm^3, where $V(x)$ is a function in *x*.

(i) Find $V(x)$. (2 marks)

(ii) Find $V'(x)$ and $V''(x)$. (2 marks)

(iii) Sketch the curve $y = V(x)$. In the graph, give the local maximum points, local minimum points and inflection points of $V(x)$. (7 marks)

(b) Find the area of the region bounded by the curve $y = x^2 + x$ and the line $y = 3 - x$. (5 marks)

10. Given the circle $C: \ x^2 + y^2 - 2x + 4y - 4 = 0$. Suppose that the straight line $L: y = x + b$ intersects with C at two distinct points $A(x_1, y_1)$ and $B(x_2, y_2)$.

(a) Find the center and radius of C. (2 marks)

(b) (i) Show that x_1 and x_2 satisfy the equation

$$2x^2 + (2b + 2)x + (b^2 + 4b - 4) = 0.$$ (1 marks)

(ii) Express $x_1 + x_2$, $x_1 x_2$, $y_1 + y_2$ and $y_1 y_2$ in terms of b. (4 marks)

(c) Find the distance between A and B. Give your answer in terms of b. (5 marks)

(d) Suppose that the circle having the segment AB as a diameter passes through the origin O. Find the value(s) of b. (4 marks)

11. Let $i = \sqrt{-1}$.

(a) Suppose z is a complex number and satisfies $4z + 2\bar{z} = 3\sqrt{3} + i$.

(i) Find z, and express z in polar form $r(\cos\alpha + i\sin\alpha)$, $-\pi < \alpha \le \pi$. (5 marks)

(ii) Let $u = \sin\theta + i\cos\theta$, $-\pi < \theta \le \pi$. Find the range of $|z - u|$. (4 marks)

(b) Suppose w is a complex number and satisfies $w^7 = 1$, $w \neq 1$. Let β be the argument of w.

(i) Show that $1 + w + w^2 + w^3 + w^4 + w^5 + w^6 = 0$. (1 marks)

(ii) Show that for any positive integer n, $w^n + w^{-n} = 2\cos(n\beta)$. (3 marks)

(iii) Find the value of $\cos\beta + \cos(2\beta) + \cos(3\beta)$. (3 marks)

12. (a) Factorize the determinant $\begin{vmatrix} a^2 & b^2 & c^2 \\ (1+a)^2 & (1+b)^2 & (1+c)^2 \\ (2+a)^2 & (2+b)^2 & (2+c)^2 \end{vmatrix}$. (6 marks)

(b) Given a system of equation (E) with unknowns x, y and z:

$$(E)\begin{cases} x + py + pz = 1 \\ px + y + pz = q \\ px + py + z = 1 \end{cases}.$$

(i) Find the values of p such that (E) has a unique solution. (4 marks)

(ii) Find the general solution of (E) for those values of p and q such that (E) has more than one solution. (6 marks)

End of Paper

2015/2016 學年 參考答案 MODEL ANSWER

1. (a) 40

   (b) $n = 11$

2. (a) $\{x: 1 < x < 7\}$

   (b) $\{ x: x > \frac{1}{\sqrt{\ln 10}}\}$

3. (a) For LHS = 1 = RHS.

   Suppose the result is true for $n = k$. Then

   $$1^3 + \cdots + k^3 + (k+1)^3 = \frac{1}{4}k^2(k+1)^2 + (k+1)^3 = \frac{1}{4}(k+1)^2(k+2)^2.$$

   Hence the result is true for $n = k+1$. By the principle of mathematical induction, the result is true for all positive integers.

   (b) $-n^2(4n+3)$

4. (a) $\cos\frac{A-B}{2} = \frac{\sqrt{2}}{2} - \sqrt{2}\cos(A-B)$

   (b) $\frac{\sqrt{2}}{2}$

5. $1 < x < \log_2 3$ or $x > \log_2 7$

6. (a) $a_n = 13 - 4n$

   (b) Write $S_n = |a_1| + \cdots + |a_n|$. Then $S_1 = 9$, $S_2 = 14$, $S_3 = 15$, and

   $S_n = 2n^2 - 11n + 30$ for $n \geq 4$.

7. (a) $\frac{5}{42}$

   (b) $\frac{1}{4}$

8. (a) Given ΔPDC is equilateral and so $PE \perp DC$. With $PDC \perp ABCD$, we get $PE \perp ABCD$. Hence, the angle between *PA* and *ABCD* is $\angle PAE$.

   From ΔPDC, $|PE| = 2\sin\frac{\pi}{3} = \sqrt{3}$.

   From ΔDEA, $|AE|^2 = |AD|^2 + |DE|^2 - 2|AD||DE|\cos\frac{\pi}{3} = 3$. Hence, $|AE| = \sqrt{3}$.

   Thus, $\angle PAE = \tan^{-1}\frac{|PE|}{|AE|} = \frac{\pi}{4}$.

(b) (i) In ΔDEA, $|DE|^2+|AE|^2=1^2+(\sqrt{3})^2=2^2=|AD|^2$ and so $\angle DEA=\frac{\pi}{2}$. Thus,

$CE\perp EA$. Together with $CE\perp EP$, we have $CE\perp \Delta PEA$ and hence $\angle CEN=\frac{\pi}{2}$.

In ΔPAB, as M and N are midpoints of PB and PA, respectively, we get $MN\parallel AB$. In rhombus $ABCD$, $AB\parallel CD$. Hence $MN\parallel CE$. In ΔPAB, we have $|MN|=\frac{1}{2}|AB|=1$. Together with $|EC|=1$, we have that $MNEC$ is a rectangle.

From (a), $|PE|=|EA|$, i.e., ΔEPA is isosceles. As N is the midpoint of PA, we have $EN\perp PA$. From above, $MN\parallel CE$ and $CE\perp \Delta PEA$, we have $MN\perp PA$. Thus, $PA\perp MNEC$.

(ii) We have $|MC|=|NE|=|AE|\sin\angle NAE=\sqrt{3}\sin\frac{\pi}{4}=\frac{\sqrt{6}}{2}$. Hence the area of

ΔDCM is $\frac{1}{2}|MC||DC|=\frac{\sqrt{6}}{2}$. We also have $|NP|=|NA|=|AE|\cos\angle NAE=$

$\sqrt{3}\cos\frac{\pi}{4}=\frac{\sqrt{6}}{2}$. Hence the volume is $\frac{1}{3}\times(\text{area of }\Delta DCM)\times|NP| =\frac{1}{2}$.

9. (a) (i) $V(x)=\pi(36x^2-x^3)$, $0\le x\le 36$.

(ii) $V'(x)=3\pi(24x-x^2)$, $V''(x)=6\pi(12-x)$

(iii) The curve $y=V(x)$ attains a local maximum point at $x=24$, and an inflection point at $x=12$. It is increasing on [0,24], decreasing on [24,36], convex on [0,12] and concave on [12,36].

(b) $\int_{-3}^{1}(3-x)-(x^2+x)\,dx=\dfrac{32}{3}$

10. (a) (i) Center is (1,-2), radius is 3

(b) (i) Putting $y=x+b$ in the equation $x^2+y^2-2x+4y-4=0$, the result follows.

(ii) $x_1+x_2=-b-1$, $x_1x_2=\dfrac{b^2+4b-4}{2}$, $y_1+y_2=(x_1+b)+(x_2+b)=b-1$,

$y_1y_2=(x_1+b)(x_2+b)=\dfrac{b^2+2b-4}{2}$.

(c)

$$\begin{aligned}
|AB| &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= \sqrt{(x_1^2 + y_1^2) + (x_2^2 + y_2^2) - 2x_1x_2 - 2y_1y_2} \\
&= \sqrt{(2x_1 - 4y_1 + 4) + (2x_2 - 4y_2 + 4) - 2x_1x_2 - 2y_1y_2} \\
&= \sqrt{2(x_1 + x_2) - 4(y_1 + y_2) + 8 - 2x_1x_2 - 2y_1y_2} \\
&= \sqrt{18 - 12b - 2b^2}
\end{aligned}$$

(d) We have $OA \perp OB$ and so $\frac{y_1}{x_1} \cdot \frac{y_2}{x_2} = -1$, i.e., $x_1x_2 + y_1y_2 = 0$ Thus,

$\frac{b^2 + 4b - 4}{2} + \frac{b^2 + 2b - 4}{2} = 0$. Solving, we get $b = 1$ or $b = -4$.

11. (a) (i) $z = \frac{\sqrt{3}}{2} + i\frac{1}{2} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$

(ii) We have $|z - w| = \sqrt{(\cos\frac{\pi}{6} - \sin\theta)^2 + (\sin\frac{\pi}{6} - \cos\theta)^2} = \sqrt{2 - 2\sin(\theta + \frac{\pi}{6})}$

and thus $0 \le |z - w| \le 2$.

(b) (i) As $w \ne 1$,

$$w^7 - 1 = 0 \Rightarrow (w-1)(w^6 + w^5 + \cdots + 1) = 0 \Rightarrow w^6 + w^5 + \cdots + 1 = 0.$$

(ii) As $w^7 = 1$, we get $|w|^7 = 1$ and hence $|w| = 1$.

Let $w = \cos\beta + i\sin\beta$. Then $w^n = \cos(n\beta) + i\sin(n\beta)$ and also $w^{-n} = \cos(n\beta) - i\sin(n\beta)$. The result follows.

(iii) Note that as $w^7 = 1$, we have $w^6 = w^{-1}$, $w^5 = w^{-2}$ and $w^4 = w^{-3}$. Thus,

$$\begin{aligned}
& w^6 + w^5 + \cdots + 1 = 0 \\
\Rightarrow\ & (w + w^{-1}) + (w^2 + w^{-2}) + (w^3 + w^{-3}) = -1 \\
\Rightarrow\ & 2\cos\beta + 2\cos(2\beta) + 2\cos(3\beta) = -1 \\
\Rightarrow\ & \cos\beta + \cos(2\beta) + \cos(3\beta) = -\tfrac{1}{2}
\end{aligned}$$

12. (a) $4(a-b)(a-c)(b-c)$

(b) (i) We need $\begin{vmatrix} 1 & p & p \\ p & 1 & p \\ p & p & 1 \end{vmatrix} \ne 0$. Thus, $p \ne 1$ and $p \ne -\frac{1}{2}$.

(ii) When $p = 1$, we must have $q = 1$ and the system of equations becomes $x + y + z = 1$. The general solution is $x = 1 - s - t,\ y = s,\ z = t$ where s and t are any real numbers. When $p = -\frac{1}{2}$, we must have $q = -2$ (by considering the sum of all the three equations). The system of equations becomes

$$\begin{cases} x - \frac{1}{2}y - \frac{1}{2}z = 1 \\ -\frac{1}{2}x + y - \frac{1}{2}z = -2 \end{cases}$$

and the general solution is $x = t,\ y = t - 2,\ z = t$ where t is any real number.

2014/2015 學年 入學考試試題

第一部分：七題全部作答。

1. 解 $\log_x 27 - 4\log_9 x = 1$。 (6 分)

2. 設 $f(x) = \dfrac{e^{2x}-1}{e^{2x}+1}$。
   (a) 證明 $f(x)$ 在定義域上是增函數。 (2 分)
   (b) 求 $f(x)$ 的值域。 (3 分)
   (c) 求 $\arccos\left(\dfrac{|f(x)|}{2}\right)$的值域。 (3 分)

3. 設 $\{a_n\}_{n=1}^{\infty}$ 為一正項數列，使得對任意正整數 n，有

   $$a_1 + \cdots + a_n = \frac{(a_n+1)^2}{4} - 1$$ 。

   用數學歸納法，證明對任意正整數 n，有 $a_n = 2n+1$。 (8 分)

4. (a) 設 k 為正整數，求 $\cos[(k+\frac{1}{6})\pi]\tan[(3k-\frac{1}{4})\pi]$ 的值。 (3 分)
   (b) 求 $\sqrt{3}\sin 2x + \cos 2x = \sqrt{2}$ 的通解。 (5 分)

5. 解不等式 $|x^2 - 4x + 2| < 1$。 (6 分)

6. 設 $\{a_n\}_{n=1}^{\infty}$ 為一等比數列，其公比 $r > 1$，且 $a_1 + a_2 + a_3 = 21$ 及

   $a_1 a_2 a_3 = 64$ 。

   (a) 求此等比數列的通項 a_n。 (5 分)

(b) 求最小的 m 使得 $a_1+a_2+\cdots+a_m\geq 654321$。 (3 分)

7. 一袋內有红球 6 個及白球 4 個。

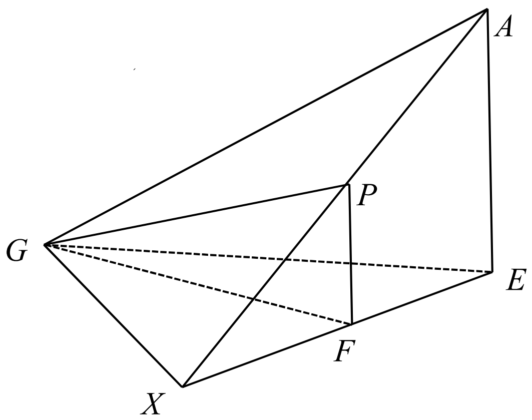
(a) 從袋中隨機逐一把球抽出，直至 4 個白球都被抽出。問第 8 次抽球剛好把 4 個白球都抽出的概率是多少? (4 分)

(b) 從袋中隨機逐一把球抽出，當中若抽出的是红球，就馬上把它放回袋內，然後再繼續抽球，直至 4 個白球都被抽出。問第 5 次抽球剛好把 4 個白球都抽出的概率是多少? (4 分)

第二部分:任擇三題作答，每題十六分。

8. (a)

圖 I



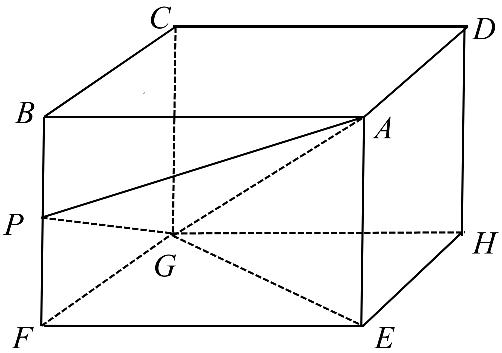
如圖 I，在三棱錐 $AEXG$ 中，$AE\perp\Delta EXG$，P 為 AX 上一點，F 為 P 在 XE 的垂足。

(i) 證明 $PF\perp\Delta EXG$。 (2 分)

(ii) 設平面 GAX 與平面 GEX 所夾的二面角為 α。證明 $S_{\Delta GEF}=S_{\Delta GAP}\cos\alpha$， 其中 $S_{\Delta GAP}$ 及 $S_{\Delta GEF}$ 分別代表 ΔGAP 及 ΔGEF 的面積。 (7 分)

(b)

圖 II



如圖 II，*ABCD-EFGH* 為一正立方體，P 為 BF 的中點。用 (a)(ii) 的結果，或用其他方法，求平面 GAP 與平面 GEF 所夾的二面角。 (7 分)

9. (a) 已知函數 $f(x)=x^3-6x^2+9x+2$。

(i) 求 $f'(x)$ 及 $f''(x)$。 (2 分)

(ii) 求 $f(x)$ 的局部極大點、局部極小點和拐點。 (4 分)

(iii) 繪出曲線 $y=f(x)$。 (2 分)

(iv) 繪出曲線 $y=f(|x-1|)$。 (2 分)

(b) 求在第一、二像限內由 x-軸，直線 $y=x$ 及曲線 $y=6-x^2$ 所包圍的區域之面積。 (6 分)

10. 已知拋物線 $P: y^2=a(x+1)$, $a>0$, 與直線 $L: x+y=b$ 交於不同的兩點 $H(x_1,y_1)$ 和 $K(x_2,y_2)$。

(a) (i) 求以 x_1 和 x_2 為根的二次方程。 (1 分)

(ii) 證明 $a+4b+4>0$。 (2 分)

(iii) 證明 $x_1x_2=b^2-a$ 及 $y_1y_2=-a-ab$。 (4 分)

(b) 設直線 OH 與直線 OK 互相垂直，其中 O 為原點。

(i) 證明 $a=\dfrac{b^2}{b+2}$ 。 (3 分)

(ii) 若直線 L 與原點 O 的距離為 a，求 b。 (6 分)

11. 設 $i=\sqrt{-1}$ 及 $-\pi\le\theta\le\pi$。

(a) (i) 按極式 $r(\cos\alpha+i\sin\alpha)$ 表示 $1+\cos\theta+i\sin\theta$。 (4 分)

(ii) 由此，簡化 $\dfrac{(1+\cos\theta+i\sin\theta)^8}{\cos\theta-i\sin\theta}$ 為極式。 (3 分)

(b) (i) 用棣美弗定理，或其他方法，證明

$\cos 7\theta=64\cos^7\theta-112\cos^5\theta+56\cos^3\theta-7\cos\theta$ 。 (3 分)

(ii) 用 (i) 的結果，證明 $64x^6-112x^4+56x^2-7=0$ 的根為

$\pm\cos\dfrac{\pi}{14},\ \pm\cos\dfrac{3\pi}{14},\ \pm\cos\dfrac{5\pi}{14}$ 。 (3 分)

(iii) 由 (ii) 的結果，求 $\cos\dfrac{\pi}{14}\cos\dfrac{3\pi}{14}\cos\dfrac{5\pi}{14}$ 的值。 (3 分)

12. (a) 因式分解行列式 $\begin{vmatrix} a^2 & (b+c)^2 & bc \\ b^2 & (c+a)^2 & ca \\ c^2 & (a+b)^2 & ab \end{vmatrix}$。 (7 分)

(b) (i) 求以 X, Y, Z 為變量的方程組 $\begin{cases} Z+Y & =a \\ Z \quad +X=b \\ \quad Y+X=c \end{cases}$ 的解。 (3 分)

(ii) 設 a, b, c 為正數。證明以 x, y, z 為變量的方程組 $\begin{cases} xy+xz & =a \\ xy \quad +yz=b \\ \quad xz+yz=c \end{cases}$

有解當且僅當 $(a+b-c)>0,\ (a+c-b)>0,\ (b+c-a)>0$。

(6 分)

全卷完

2014/2015 ADMISSION EXAMINATION PAPER

Section I. Answer all 7 questions.

1. Solve $\log_x 27 - 4\log_9 x = 1$. (6 marks)

2. Let $f(x) = \dfrac{e^{2x}-1}{e^{2x}+1}$.
   (a) Show that $f(x)$ is an increasing function on its domain. (2 marks)
   (b) Find the range of $f(x)$. (3 marks)
   (c) Find the range of $\arccos\left(\dfrac{|f(x)|}{2}\right)$. (3 marks)

3. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that for any positive integer *n*,

   $a_1 + \cdots + a_n = \dfrac{(a_n+1)^2}{4} - 1$. Use mathematical induction to show that, for any

   positive integer *n*, $a_n = 2n+1$. (8 marks)

4. (a) Let *k* be a positive integer. Find the value of

   $\cos[(k+\frac{1}{6})\pi]\tan[(3k-\frac{1}{4})\pi]$. (3 marks)

   (b) Find the general solution of $\sqrt{3}\sin 2x + \cos 2x = \sqrt{2}$. (5 marks)

5. Solve the inequality $|x^2 - 4x + 2| < 1$. (6 marks)

6. Let $\{a_n\}_{n=1}^{\infty}$ be a geometric sequence with common ratio $r > 1$. Suppose

   $a_1 + a_2 + a_3 = 21$ and $a_1 a_2 a_3 = 64$.

   (a) Find the general term a_n of the sequence. (5 marks)

   (b) Find the smallest m such that $a_1 + a_2 + \cdots + a_m \geq 654321$. (3 marks)

7. A bag contains 6 red balls and 4 white balls.

   (a) Balls are drawn randomly one by one from the bag until all the 4 white balls are drawn. What is the probability that all the four white balls are drawn at the 8th draw?

   (4 marks)

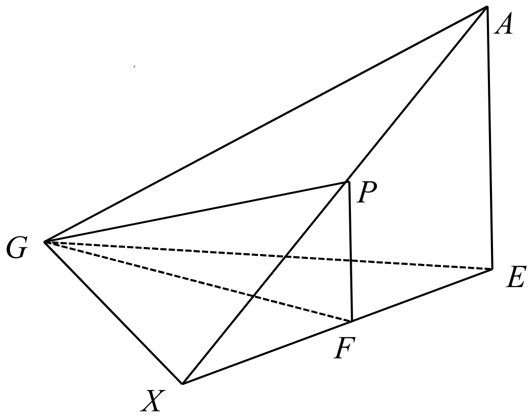
   (b) Balls are drawn randomly one by one from the bag, until all the 4 white balls are drawn. If a red ball is drawn, it is returned to the bag before the next draw. What is the probability that all the four white balls are drawn at the 5th draw?

   (4 marks)

Section II Answer any three question. Each question carries 16 marks.

8. (a)

Figure I



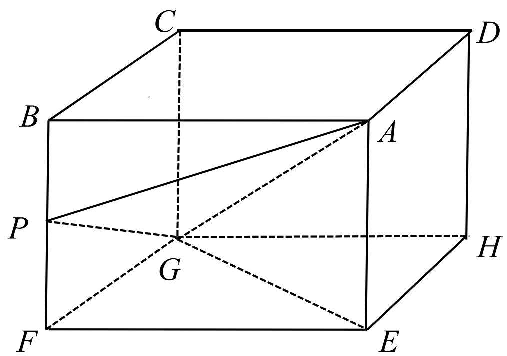
In Figure I, *AEXG* is a triangular pyramid with $AE \perp \Delta EXG$. *P* is a point on *AX*, *F* is the foot of perpendicular from *P* to *XE*.

(i) Show that $PF \perp \Delta EXG$. (2 marks)

(ii) Suppose the angle between the planes *GAX* and *GEX* is α. Show that $S_{\Delta GEF} = S_{\Delta GAP} \cos\alpha$, where $S_{\Delta GAP}$ and $S_{\Delta GEF}$ denote the area of ΔGAP and the area of ΔGEF, respectively. (7 marks)

(b)

Figure II



In Figure II, *ABCD-EFGH* is a cube. *P* is the mid-point of *BF*. Using the result in (a)(ii),

or otherwise, find the angle between the planes *GAP* and *GEF*. (7 marks)

9. (a) Given function $f(x) = x^3 - 6x^2 + 9x + 2$.

   (i) Find $f'(x)$ and $f''(x)$. (2 marks)

   (ii) Find the local maximum, local minimum and inflection points of $f(x)$

   (4 marks)

   (iii) Sketch the curve $y = f(x)$. (2 marks)

   (iv) Sketch the curve $y = f(|x-1|)$. (2 marks)

   (b) Find the area of the region in the first and second quadrants bounded by the x-axis, the line $y = x$ and the curve $y = 6 - x^2$. (6 marks)

10. Given that the parabola $P: y^2 = a(x+1)$, $a > 0$, and the straight line $L: x + y = b$ intersect at two distinct points $H(x_1, y_1)$ and $K(x_2, y_2)$.

   (a) (i) Find a quadratic equation whose roots are x_1 and x_2. (1 marks)

   (ii) Show that $a + 4b + 4 > 0$. (2 marks)

   (iii) Show that $x_1 x_2 = b^2 - a$ and $y_1 y_2 = -a - ab$. (4 marks)

   (b) Suppose the lines *OH* and *OK* are perpendicular, where *O* is the origin.

   (i) Show that $a = \dfrac{b^2}{b+2}$. (3 marks)

   (ii) If the distance between the line *L* and the origin *O* is *a*, find *b*.

   (6 marks)

11. Let $i = \sqrt{-1}$ and $-\pi \le \theta \le \pi$.

   (a) (i) Express $1 + \cos\theta + i\sin\theta$ in polar form $r(\cos\alpha + i\sin\alpha)$ (4 marks)

   (ii) Hence, simplify $\dfrac{(1+\cos\theta + i\sin\theta)^8}{\cos\theta - i\sin\theta}$ to polar form. (3 marks)

   (b) (i) Using De Movire's theorem, or otherwise, show that

   $\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$. (3 marks)

(ii) Using the result in (i), show that the roots of

$64x^6 - 112x^4 + 56x^2 - 7 = 0$ are

$$\pm\cos\frac{\pi}{14},\ \pm\cos\frac{3\pi}{14},\ \pm\cos\frac{5\pi}{14}.$$ (3 marks)

(iii) Using the result in (ii), find the value of $\cos\frac{\pi}{14}\cos\frac{3\pi}{14}\cos\frac{5\pi}{14}$.

(3 marks)

12. (a) Factorize the determinant $\begin{vmatrix} a^2 & (b+c)^2 & bc \\ b^2 & (c+a)^2 & ca \\ c^2 & (a+b)^2 & ab \end{vmatrix}$. (7 marks)

(b) (i) Find the solution of the system of equations $\begin{cases} Z+Y \qquad = a \\ Z \qquad + X = b \\ \qquad Y + X = c \end{cases}$ in

variables X,Y, Z. (3 marks)

(ii) Suppose a, b, c are positive numbers. Show that system of equations

$\begin{cases} xy + xz \qquad = a \\ xy \qquad + yz = b \\ \qquad xz + yz = c \end{cases}$ in variables x, y, z has a solution if and only if

$(a+b-c) > 0,\ (a+c-b) > 0,\ (b+c-a) > 0.$ (6 marks)

End of Paper

2014/2015 學年 參考答案 MODEL ANSWER

1. $x=3$ or $x=3^{-3/2}$

2. (a) Suppose $u>v$. Then $f(u)-f(v)=\dfrac{2(e^{2u}-e^{2v})}{(e^{2u}+1)(e^{2v}+1)}>0$.

   (b) $\{y:-1<y<1\}$

   (c) $\left\{z:\dfrac{\pi}{3}<z\le\dfrac{\pi}{2}\right\}$

3. For $n=1$, we have $a_1=\dfrac{(a_1+1)^2}{4}-1$. Solving, since a_1 is positive,

   $a_1=3=2(1)+1$.

   Thus the result is true for $n=1$.

   Suppose the result is true for $n=1,...,k$. Then

$$a_1+\cdots+a_k+a_{k+1}=\frac{(a_{k+1}+1)^2}{4}-1$$
$$\Rightarrow[3+\cdots+(2k+1)]+a_{k+1}=\frac{(a_{k+1}+1)^2}{4}-1$$
$$\Rightarrow 2\frac{k(k+1)}{2}+k+a_{k+1}=\frac{(a_{k+1}+1)^2}{4}-1$$
$$\Rightarrow a_{k+1}^2-2a_{k+1}-(4k^2+8k+3)=0$$

   Solving, as a_{k+1} is positive, we get $a_{k+1}=2(k+1)+1$. The result follows.

4. (a) $(-1)^{k+1}\dfrac{\sqrt{3}}{2}$

   (b) $\dfrac{k\pi}{2}+(-1)^k\dfrac{\pi}{8}-\dfrac{\pi}{12}$, where k is an integer

5. $2-\sqrt{3}<x<1$ or $3<x<2+\sqrt{3}$

6. (a) $a_n=4^{n-1}$

   (b) 11

7. (a) $\dfrac{1}{6}$

   (b) $\dfrac{1207}{88200}$

8. (a) (i) In ΔAXE, $AE // PF$ because $\angle PFX=\angle AEX=\dfrac{\pi}{2}$. With $AE\perp\Delta EXG$, the result follows.

(ii) Let M be the point on GX such that $\angle AME = \alpha$. Then $AM \perp GX$,

$EM \perp GX$ and $S_{\Delta EGX} = \frac{1}{2}|EM||GX| = \frac{1}{2}(|AM|\cos\alpha)|GX| = S_{\Delta AGX}\cos\alpha$.

Similarly, let N be the point on GX such that $\angle PNF = \alpha$. Then $PN \perp GX$, $FN \perp GX$ and

$S_{\Delta FGX} = \frac{1}{2}|FN||GX| = \frac{1}{2}(|PN|\cos\alpha)|GX| = S_{\Delta PGX}\cos\alpha$. Hence,

$S_{\Delta GEF} = S_{\Delta EGX} - S_{\Delta FGX} = S_{\Delta AGX}\cos\alpha - S_{\Delta PGX}\cos\alpha = S_{\Delta GAP}\cos\alpha$.

(b) Suppose the length of one side of the cube is a. Then, we can get $S_{\Delta GEF} = \frac{1}{2}a^2$ and $S_{\Delta GAP} = \sqrt{\frac{3}{8}}a^2$. Hence, the required angle is $\cos^{-1}\sqrt{\frac{2}{3}}$.

9. (a) (i) $f'(x) = 3x^2 - 12x + 9$, $f''(x) = 6x - 12$

(ii) & (iii) The curve $y = f(x)$ attains a local maximum point at $x = 1$, a local minimum point at $x = 3$, and an inflection point at $x = 2$.

(iv) Replace the curve $y = f(x)$, $x < 1$, by the mirror image of $y = f(x)$, $x > 1$ about the line $x = 1$.

(b) $\int_{-\sqrt{6}}^{0} 6 - x^2\,dx + \int_0^2 6 - x^2 - x\,dx = 4\sqrt{6} + \frac{22}{3}$

10. (a) (i) $x^2 - (a+2b)x + (b^2 - a) = 0$

(ii) The equation in (i) has real and distinct roots. Its discriminant is positive, i.e., $[-(a+2b)]^2 - 4(1)(b^2 - a) > 0$. The result follows.

(ii) Product of roots: $x_1x_2 = \frac{b^2 - a}{1} = b^2 - a$;

$$\begin{aligned} y_1y_2 &= (b - x_1)(b - x_2) \\ &= b^2 - b(x_1 + x_2) + x_1x_2 \\ &= b^2 - b(a+2b) + (b^2 - a) \\ &= -a - ab \end{aligned}$$

(b) (i) Suppose $x_1x_2 \neq 0$. As $OH \perp OK$, we have $\frac{y_1y_2}{x_1x_2} = -1$. The result follows from (a) (ii). If $x_1x_2 = 0$, then by (a) (ii), $a = b^2$. Also, OH and OK are the two axes and so H or K is the point $(-1,0)$. Hence $b = -1$ and the result follows.

(b) (ii) Distance from O to L is $\dfrac{|0+0-b|}{\sqrt{1^2+1^2}}=\dfrac{|b|}{\sqrt{2}}$. Hence we have $|b|=\sqrt{2}a$.

Using (i), we get $b=2+2\sqrt{2}$ if $b>0$; and $b=2-2\sqrt{2}$ if $b<0$.

11. (a) (i) In the question, θ should satisfy $-\pi<\theta<\pi$.

Let $1+\cos\theta+i\sin\theta=r(\cos\alpha+i\sin\alpha)$ where $r>0$. Then,

$r=\sqrt{(1+\cos\theta)^2+\sin^2\theta}=\sqrt{2(1+\cos\theta)}=2\cos\dfrac{\theta}{2}$ (as $-\dfrac{\pi}{2}<\dfrac{\theta}{2}<\dfrac{\pi}{2}$)and

$\tan\alpha=\dfrac{\sin\theta}{1+\cos\theta}=\dfrac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}=\tan\dfrac{\theta}{2}$. Hence, $\alpha=\dfrac{\theta}{2}$ and thus

$1+\cos\theta+i\sin\theta=2\cos\dfrac{\theta}{2}\left(\cos\dfrac{\theta}{2}+i\sin\dfrac{\theta}{2}\right)$.

(b)

$$\frac{(1+\cos\theta+i\sin\theta)^8}{\cos\theta-i\sin\theta}=\frac{\left[2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)\right]^8}{\cos(-\theta)+i\sin(-\theta)}$$
$$=\frac{2^8\cos^8\frac{\theta}{2}(\cos4\theta+i\sin4\theta)}{\cos(-\theta)+i\sin(-\theta)}$$
$$=2^8\cos^8\frac{\theta}{2}(\cos5\theta+i\sin5\theta)$$

(b) (i) By comparing the real parts of both sides of $(\cos\theta+i\sin\theta)^7=\cos7\theta+i\sin7\theta$, the result follows.

(ii) Putting $x=\cos\theta$, we have

$64x^6-112x^4+56x^2-7=0$

$\Leftrightarrow 64\cos^6\theta-112\cos^4\theta+56\cos^2\theta-7=0$

$\Leftrightarrow \cos7\theta=0$ and $\cos\theta\neq0$

For $\cos7\theta=0$, $\theta=\dfrac{k\pi}{7}+\dfrac{\pi}{14}$, where k is an integer. Excluding those θ such that $\cos\theta=0$, the result follows.

(iii) Product of roots is $\dfrac{-7}{64}$ and hence $-\cos^2\dfrac{\pi}{14}\cos^2\dfrac{3\pi}{14}\cos^2\dfrac{5\pi}{14}=\dfrac{-7}{64}$.

The result follows.

12. (a) $-(a^2+b^2+c^2)(a+b+c)(a-b)(b-c)(c-a)$

(b) (i) $X=\dfrac{b+c-a}{2}$, $Y=\dfrac{a+c-b}{2}$, $Z=\dfrac{a+b-c}{2}$

(ii) From (i), we have $yz=\dfrac{b+c-a}{2}$, $xz=\dfrac{a+c-b}{2}$, $xy=\dfrac{a+b-c}{2}$.

Suppose $a+b-c>0$, $a+c-b>0$ and $b+c-a>0$. Then,

$$x^2=\frac{(xy)(xz)}{(yz)}=\frac{(a+b-c)(a+c-b)}{2(b+c-a)}.(*)$$

Hence $x=\sqrt{\dfrac{(a+b-c)(a+c-b)}{2(b+c-a)}}$, (and similarly)

$$y=\sqrt{\frac{(a+b-c)(b+c-a)}{2(a+c-b)}}$$

and $z=\sqrt{\dfrac{(a+c-b)(b+c-a)}{2(a+b-c)}}$ form a solution.

Conversely, if there is a solution then *x, y, z* are nonzero. From (*), either 0 or 2 of the factors on the right-hand side are negative. If there are 2 negative factors, say, $a+b-c<0$ and $a+c-b<0$, then

$a=xy+xz=\dfrac{a+b-c}{2}+\dfrac{a+c-b}{2}<0$ and this gives a contradiction. Hence all the 3 factors are positive.