澳門大學 UNIVERSIDADE DE MACAU UNIVERSITY OF MACAU

澳門理工學院 Instituto Politécnico de Macau Macao Polytechnic Institute

旅遊學院 INSTITUTO DE FORMAÇÃO TURÍSTICA Institute for Tourism Studies

澳門科技大學 UNIVERSIDADE DE CIÊNCIA E TECNOLOGIA DE MACAU MACAU UNIVERSITY OF SCIENCE AND TECHNOLOGY

澳門四高校聯合入學考試 (語言科及數學科)

Joint Admission Examination for Macao Four Higher Education Institutions (Languages and Mathematics)

模擬試題及參考答案 Mock Paper and Suggested Answer

數學正卷 Mathematics Standard Paper

指示：

1. 本卷分為兩部份：第一部份有十五條選擇題，第二部份有五條解答題。**全部為必答題**。

2. 第一部份有十五條選擇題，每題佔四分，共佔六十分。本部份答案必須使用 HB 鉛筆作答。

3. 第二部份有五條解答題。全部為必答題，共佔四十分。每題所佔分數在題末註明。答案必須使用藍色或黑色原子筆在每題之指定位置作答。考生必須將解題步驟清楚寫出，只當答案和所有步驟正確及清楚地表示出來，方可獲得滿分。

4. 寫在其他地方的答案或不按指示作答之考生將不會獲評分。

<u>**第一部份**</u> **選擇題。請選出每題之*最佳答案*。**

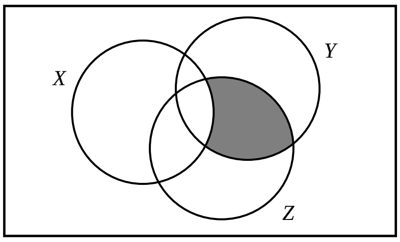
1. 當 x 由 $-\frac{\pi}{4}$ 增加至 $\frac{3\pi}{4}$，$\cos x$ 的值

A. 一直上升　　B. 一直下降　　C. 先上升後下降
D. 先下降後上升　　E. 以上皆非

2. 某手表的標價比成本高出 40% 。若該手表以七折出售，求盈利或虧蝕百分率。

A. 賺 28%　　B. 賺 12%　　C. 賺 10%　　D. 蝕 2%　　E. 蝕 10%

3. 在偉恩圖中，陰影部份表示



A. $X\cup(Y\cap Z)$　　B. $X^c\cup(Y\cap Z)$　　C. $X\cup(Y\cap Z)^c$
D. $(X\cap Y)^c\cap Z$　　E. $(X^c\cap Y)\cap Z$

4. 若 $y\propto x$，下列何者為正確?

I. $x\propto y$　　II. $y^2\propto x^2$　　III. $(y+x)\propto 2x$

A. 只有 II　　B. 只有 III　　C. 只有 I 和 II
D. 只有 I 和 III　　E. I、II 和 III

5. 已知多項式 $p(x)$ 可被 $2x+1$ 整除，而被 $x-1$ 除餘數為 1。若 $p(x)$ 被 $2x^2-x-1$ 除，餘式是甚麼？

A. $\frac{2}{3}x+\frac{1}{3}$　　B. $x+\frac{1}{3}$　　C. $\frac{1}{3}x+\frac{2}{3}$　　D. $3x+\frac{1}{3}$　　E. $2x+\frac{2}{3}$

6. 若 a 和 b 為方程 $2x^2-x-5=0$ 的兩個不相同的根，則 $\frac{1}{a^2}+\frac{1}{b^2}=$

A. $\frac{25}{19}$　　B. $\frac{19}{25}$　　C. $\frac{37}{44}$　　D. $\frac{21}{25}$　　E. $\frac{23}{25}$

7. 若 $x+\frac{1}{x}=7$ ，則 $\sqrt{x}-\frac{1}{\sqrt{x}}=$

A. $\pm\sqrt{7}$　　B. $\pm\sqrt{5}$　　C. ±3　　D. ±7　　E. 以上皆非

8. 若 $a<2$ 及 $ax+2<2x-3$, 則

A. $x<\frac{-5}{a+2}$ B. $x<\frac{5}{a-2}$ C. $x>\frac{5}{a-2}$ D. $x>\frac{5}{2-a}$ E. $x<\frac{5}{a+2}$

9. 若 $\frac{1}{2}\log_{10} y=1+\log_{10} x$，則

A. $y=\sqrt{10x}$ B. $y=100x^2$ C. $y=(10+x)^2$ D. $y=10x^2$ E. 以上皆非

10. $\frac{2}{x^2-1}-\frac{x-1}{x^2-2x-3}=$

A. $\frac{-x^2+2x+5}{(x-1)(x+1)(x+3)}$ B. $\frac{-x^2+2x+7}{(x-1)(x+1)(x+3)}$ C. $\frac{-x^2-5}{(x-3)(x-1)(x+1)}$

D. $\frac{x^2-5}{(x-3)(x-1)(x+1)}$ E. $\frac{-x^2+4x-7}{(x-3)(x-1)(x+1)}$

11. 有 2 男孩及 3 女孩到課室上課，他們在課室佔了一行 5 座位。如果 2 個男孩要坐在一起，問總共有多少個可能排座的方法?

A. 12 B. 24 C. 36 D. 48 E. 以上皆非

12. 一幾何數列的各項均為正數，且任何項均等於後兩項之和，求公比之值。

A. 1 B. $\frac{\sqrt{5}}{2}$ C. $\frac{\sqrt{5}-1}{2}$ D. $\frac{1-\sqrt{5}}{2}$ E. $\frac{2}{\sqrt{5}}$

13. 半徑為 1 的圓形裏的內接正八邊形的面積是多少？

A. $\sqrt{2}$ B. $2\sqrt{2}$ C. $3\sqrt{2}$ D. $\pi\sqrt{2}$ E. $4\sqrt{2}$

14. 若 $\tan\theta=2$，其中 θ 落在第三象限，則 $\sin\theta-\cos\theta$ 的值是多少？

A. $-\frac{1}{2}$ B. $-\frac{1}{\sqrt{5}}$ C. $-\frac{3}{\sqrt{5}}$ D. $\frac{1}{\sqrt{5}}$ E. $\frac{3}{\sqrt{5}}$

15. 若兩直線 $ax+by+c=0$ 和 $px+qy+r=0$ 互相垂直，則下列何者為真？

A. $aq=bp$ B. $ap+bq=0$ C. $ac=br$

D. $ap+cr=0$ E. 以上皆非

<u>第二部份</u> 解答題。

1. 已知拋物綫 $y=-x^2+4x-3$ 與 x 軸相交於 A、B 兩點，並且與 y 軸相交於 C 點。求 ΔABC 的面積。 (8 分)

2. 把 $\dfrac{x^3}{(x-2)(x+1)}$ 化為部份分式。 (8 分)

3. A 袋有 4 個紅球、6 個黄球和 10 個綠球; B 袋 有 12 個紅球、4 個黄球和 4 個綠球。若在每個袋中隨機地抽出一個球，求以下各項的概率。
   (a) 兩個球都是紅色的。 (2 分)
   (b) 兩個球都不是紅色的。 (2 分)
   (c) 最少有一個球是紅色的。 (2 分)
   (d) 兩個球都是紅色或兩個球都不是紅色的。 (2 分)

4. (a) 利用圖解法解不等式組 $\begin{cases} x+5y \le 8 \\ 3x+y \le 17 \end{cases}$。 (4 分)
   (b) 設 $C=2x+y$，其中 (x, y) 是 (a) 中的不等式組的一個解。C 的極大值是甚麼？ (4 分)

5. (a) 設 x 和 y 為整數。證明若 $x-y$ 及 y 都可被 6 整除，那麼 x 亦可被 6 整除。 (2 分)
   (b) 利用 (a) 及數學歸納法證明對所有正整數 n，$4n^3+3n^2+5n$ 為 6 的倍數。 (6 分)

參考答案

<u>第一部份</u> 選擇題。

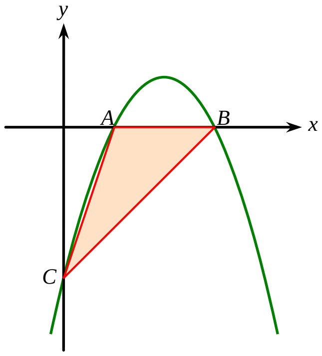
題目編號	最佳答案
1	C
2	D
3	E
4	E
5	A
6	D
7	B
8	D
9	B
10	E
11	D
12	C
13	B
14	B
15	B

(第二部分答案由下頁開始)

第二部份 解答題。

1. 已知抛物綫 $y=-x^2+4x-3$ 與 x 軸相交於 A、B 兩點，並且與 y 軸相交於 C 點。求 ΔABC 的面積。(8 分)

答案: 因為 $-x^2+4x-3=-(x-1)(x-3)$，所以抛物綫的 x 截距位於 $A(1, 0)$ 和 $B(3, 0)$。 因為該二次多項式的常數項是 -3, 所以抛物綫的 y 截距位於 $C(0, -3)$ (見下圖)。



由上可知 ΔABC 是一個底長為 2 (=3−1)、高為 3 的一個三角形。

$\therefore$ ΔABC 的面積 $=\frac{1}{2}\cdot 2\cdot 3=3$。

2. 把 $\dfrac{x^3}{(x-2)(x+1)}$ 化為部份分式。 (8 分)

答案: 利用長除法，得 $\dfrac{x^3}{(x-2)(x+1)}=x+1+\dfrac{3x+2}{(x-2)(x+1)}$。

$\dfrac{3x+2}{(x-2)(x+1)}$ 的部份分式為 $\dfrac{3x+2}{(x-2)(x+1)}=\dfrac{A}{x-2}+\dfrac{B}{x+1}$。

通過比較等式兩邊的項，得 $A=\frac{8}{3}$ 及 $B=\frac{1}{3}$。

$\therefore$ $\dfrac{x^3}{(x-2)(x+1)}=x+1+\dfrac{\frac{8}{3}}{x-2}+\dfrac{\frac{1}{3}}{x+1}=x+1+\dfrac{8}{3(x-2)}+\dfrac{1}{3(x+1)}$。

3. A 袋有 4 個紅球、6 個黃球和 10 個綠球; B 袋 有 12 個紅球、4 個黃球和 4 個綠球。若在每個袋中隨機地抽出一個球，求以下各項的概率。
(a) 兩個球都是紅色的。 (2 分)
(b) 兩個球都不是紅色的。 (2 分)
(c) 最少有一個球是紅色的。 (2 分)
(d) 兩個球都是紅色或兩個球都不是紅色的。 (2 分)

答案: (a) 從 A、B 袋抽出紅球的概率分別為 $\dfrac{4}{20}$、$\dfrac{12}{20}$。從 A 袋抽球的事件顯然獨立於從 B 袋抽球的事件。

$\therefore$ 本部份所求的概率 $=\dfrac{4}{20}\cdot\dfrac{12}{20}=\dfrac{1}{5}\cdot\dfrac{3}{5}=\dfrac{3}{25}$。

(b) 從 (a) 得本部份所求的概率 $=\left(1-\dfrac{1}{5}\right)\cdot\left(1-\dfrac{3}{5}\right)=\dfrac{8}{25}$。

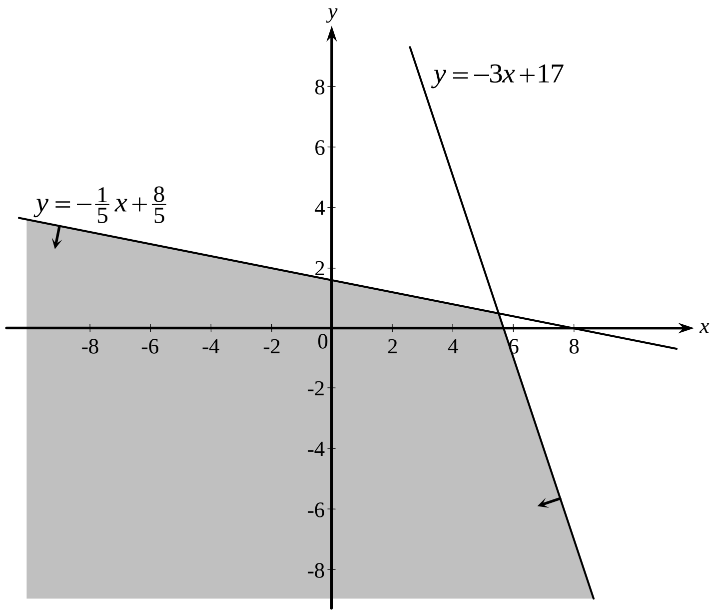
(c) 從 (b) 得本部份所求的概率 $=1-\dfrac{8}{25}=\dfrac{17}{25}$。

(d) 由於 (a) 和 (b) 中的事件互斥，本部份所求的概率 $=\dfrac{3}{25}+\dfrac{8}{25}=\dfrac{11}{25}$。

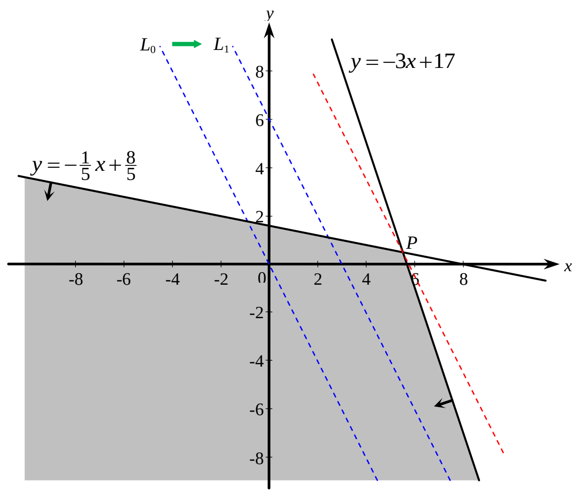
4. (a) 利用圖解法解不等式組 $\begin{cases} x+5y\le 8 \\ 3x+y\le 17 \end{cases}$。 (4 分)

(b) 設 $C=2x+y$，其中 (x, y) 是 (a) 中的不等式組的一個解。C 的極大值是甚麼？ (4 分)

答案:(a) 題中的不等式組的圖解是 $y=-\frac{1}{5}x+\frac{8}{5}$ 的下半平面和 $y=-3x+17$ 的下半平面的共同區域 (包括邊界)，如下圖中的陰影部份所示。



(b) 設 L_0 和 L_1 分別代表直綫 $2x+y=0$ 和 $2x+y=6$。從這兩條直綫的位置，可以看出將 L_0 向右推會令 C 值增加；所以應將 L_0 向右推，越遠越好，但要 L_0 觸碰着圖中的陰影區域。



從上圖可知 C 在 P 點達到極大化，這裏的 P 點是直綫 $y=-\frac{1}{5}x+\frac{8}{5}$ 和 $y=-3x+17$ 的交點。解這兩個方程式得 $P=\left(\frac{11}{2}, \frac{1}{2}\right)$。$\therefore$ C 的極大值$=2\cdot\frac{11}{2}+\frac{1}{2}=\frac{23}{2}=11.5$。

5. (a) 設 x 和 y 為整數。證明若 $x-y$ 及 y 都可被 6 整除，那麼 x 亦可被 6 整除。 (2 分)

(b) 利用 (a) 及數學歸納法證明對所有正整數 n，$4n^3+3n^2+5n$ 為 6 的倍數。 (6 分)

答案: (a) 若 $x-y$ 及 y 都可被 6 整除，那麼存在整數 m 和 n 使得 $x-y=6m$ 及 $y=6n$。

$\therefore x=(x-y)+y=6m+6n=6(m+n)$，此為 6 的倍數。証畢。

(b) 設 $S(n)$ 表示命題 "$4n^3+3n^2+5n$ 為 6 的倍數"。

(i) 由於 $4\cdot1^3+3\cdot1^2+5\cdot1=12$ 為 6 的倍數，$S(1)$ 成立。

(ii) 假設 $S(k)$ 對某正整數 k 成立，即假定

$$4k^3+3k^2+5k \text{ 可被 6 整除} \qquad \text{---------------- (1)}$$

從以下等式

$$\begin{aligned}&[4(k+1)^3+3(k+1)^2+5(k+1)]-(4k^3+3k^2+5k)\\ &=4(k^3+3k^2+3k+1)+3(k^2+2k+1)+5(k+1)-(4k^3+3k^2+5k)\\ &=6(2k^2+3k+2)\end{aligned}$$

得知

$$[4(k+1)^3+3(k+1)^2+5(k+1)]-(4k^3+3k^2+5k) \text{ 可被 6 整除} \qquad \text{---------------- (2)}$$

利用 (2)、(1) 和 (a) 就可斷定 $4(k+1)^3+3(k+1)^2+5(k+1)$ 可被 6 整除。

換句話說，$S(k+1)$ 也成立。

根據 (i)、(ii) 和數學歸納法原理，可知 $S(n)$ 對所有正整數 n 都成立。

Instructions:

1. This paper consists of 2 parts: Part I consists of 15 multiple choice questions, and Part II consists of 5 problem-solving questions. **Answer *all* the questions**.

2. Part I consists of 15 multiple choice questions. Each question carries 4 marks with a total of 60 marks. This part must be answered with HB pencil.

3. Part II consists of 5 problem-solving questions. Part II carries a total of 40 marks, and marks for each question are indicated at the end of the question. This part must be answered in the space provided for each question with blue or black ball pen. Show all your steps in getting the answer. Full credits will be given only if the answer and all the steps are correct and clearly shown.

4. Answers put elsewhere will not be marked or marks will not be given if candidates fail to follow the instructions.

<u>Part I</u> Multiple choice questions. Choose the *best answer* for each question.

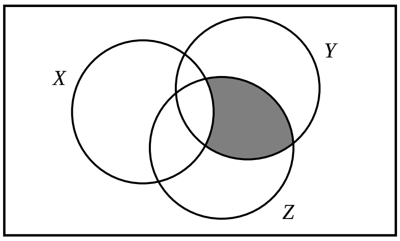
1. As x increases from $-\frac{\pi}{4}$ to $\frac{3\pi}{4}$, the value of $\cos x$

A. always increases B. always decreases C. increases and then decreases
D. decreases and then increases E. none of the above

2. The marked price of a watch is 40% above its cost. If the watch is sold at a discount of 30%, find the percentage of gain or loss.

A. gain 28% B. gain 12% C. gain 10% D. lose 2% E. lose 10%

3. In the Venn diagram, the shaded region represents



A. $X \cup (Y \cap Z)$ B. $X^c \cup (Y \cap Z)$ C. $X \cup (Y \cap Z)^c$
D. $(X \cap Y)^c \cap Z$ E. $(X^c \cap Y) \cap Z$

4. If $y \propto x$, which of the following is/are true?

I. $x \propto y$ II. $y^2 \propto x^2$ III. $(y+x) \propto 2x$

A. II only B. III only C. I and II only
D. I and III only E. I, II and III

5. It is known that a polynomial $p(x)$ is divisible by $2x+1$, and the remainder is 1 when $p(x)$ is divided by $x-1$. What is the remainder if $p(x)$ is divided by $2x^2-x-1$?

A. $\frac{2}{3}x+\frac{1}{3}$ B. $x+\frac{1}{3}$ C. $\frac{1}{3}x+\frac{2}{3}$ D. $3x+\frac{1}{3}$ E. $2x+\frac{2}{3}$

6. If a and b are the two distinct roots of the equation $2x^2-x-5=0$, then $\frac{1}{a^2}+\frac{1}{b^2}=$

A. $\frac{25}{19}$ B. $\frac{19}{25}$ C. $\frac{37}{44}$ D. $\frac{21}{25}$ E. $\frac{23}{25}$

7. If $x+\frac{1}{x}=7$, then $\sqrt{x}-\frac{1}{\sqrt{x}}=$

A. $\pm\sqrt{7}$ B. $\pm\sqrt{5}$ C. ± 3 D. ± 7 E. None of the above

8. If $a<2$ and $ax+2<2x-3$, then

A. $x<\frac{-5}{a+2}$ B. $x<\frac{5}{a-2}$ C. $x>\frac{5}{a-2}$ D. $x>\frac{5}{2-a}$ E. $x<\frac{5}{a+2}$

9. If $\frac{1}{2}\log_{10} y=1+\log_{10} x$, then

A. $y=\sqrt{10x}$ B. $y=100x^2$ C. $y=(10+x)^2$ D. $y=10x^2$ E. None of the above

10. $\frac{2}{x^2-1}-\frac{x-1}{x^2-2x-3}=$

A. $\frac{-x^2+2x+5}{(x-1)(x+1)(x+3)}$ B. $\frac{-x^2+2x+7}{(x-1)(x+1)(x+3)}$ C. $\frac{-x^2-5}{(x-3)(x-1)(x+1)}$

D. $\frac{x^2-5}{(x-3)(x-1)(x+1)}$ E. $\frac{-x^2+4x-7}{(x-3)(x-1)(x+1)}$

11. There are 2 boys and 3 girls to attend a lecture. They take a 5 seat row in the classroom. How many different possible ways of sitting are there if 2 boys are required to sit together?

A. 12 B. 24 C. 36 D. 48 E. None of the above

12. In a geometric sequence, all numbers are positive. Any term of the sequence is always equal to the sum of the next two terms. Determine the value of the common ratio.

A. 1 B. $\frac{\sqrt{5}}{2}$ C. $\frac{\sqrt{5}-1}{2}$ D. $\frac{1-\sqrt{5}}{2}$ E. $\frac{2}{\sqrt{5}}$

13. Find the area of the inscribed regular octagon in a circle of radius 1.

A. $\sqrt{2}$ B. $2\sqrt{2}$ C. $3\sqrt{2}$ D. $\pi\sqrt{2}$ E. $4\sqrt{2}$

14. If $\tan\theta=2$, where θ lies in the third quadrant, what is the value of $\sin\theta-\cos\theta$?

A. $-\frac{1}{2}$ B. $-\frac{1}{\sqrt{5}}$ C. $-\frac{3}{\sqrt{5}}$ D. $\frac{1}{\sqrt{5}}$ E. $\frac{3}{\sqrt{5}}$

15. If two lines $ax+by+c=0$ and $px+qy+r=0$ are perpendicular, which of the following is true?

A. $aq=bp$ B. $ap+bq=0$ C. $ac=br$

D. $ap+cr=0$ E. None of the above

<u>Part II</u> Problem-solving questions.

1. Suppose the parabola $y = -x^2 + 4x - 3$ intersects the x-axis at points A and B, and intersects the y-axis at point C. Determine the area of ΔABC. (8 marks)

2. Find the partial fraction decomposition of $\frac{x^3}{(x-2)(x+1)}$. (8 marks)

3. In Bag A there are 4 red balls, 6 yellow balls and 10 green balls. In Bag B there are 12 red balls, 4 yellow balls and 4 green balls. If one ball is randomly selected from each bag, find the probability that
   (a) both balls are red. (2 marks)
   (b) both balls are not red. (2 marks)
   (c) at least one ball is red. (2 marks)
   (d) both balls are red or both balls are not red. (2 marks)

4. (a) Solve the system of inequalities $\begin{cases} x + 5y \le 8 \\ 3x + y \le 17 \end{cases}$ graphically. (4 marks)
   (b) Let $C = 2x + y$, where (x, y) is a solution of the system of inequalities in (a). What is the maximum value of C? (4 marks)

5. (a) Let x and y be integers. Prove that if both $x - y$ and y are divisible by 6, so is x. (2 marks)
   (b) Use (a) and mathematical induction to show that $4n^3 + 3n^2 + 5n$ is a multiple of 6 for all positive integers n. (6 marks)

Suggested Answer

Part I **Multiple choice questions.**

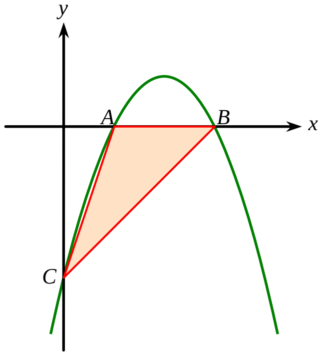
Question Number	Best Answer
1	C
2	D
3	E
4	E
5	A
6	D
7	B
8	D
9	B
10	E
11	D
12	C
13	B
14	B
15	B

(Answers for Part II start from next page)

<u>Part II</u> Problem-solving questions.

1. Suppose the parabola $y=-x^2+4x-3$ intersects the x-axis at points A and B, and intersects the y-axis at point C. Determine the area of ΔABC. (8 marks)

Ans.: Since $-x^2+4x-3=-(x-1)(x-3)$, the x-intercepts of the parabola are at $A(1, 0)$ and $B(3, 0)$. Since the constant term of the quadratic polynomial is -3, the y-intercept is at $C(0, -3)$ (see the picture below).



$\therefore$ ΔABC can be considered as a triangle of base length 2 $(=3-1)$ and of height 3.

$\therefore$ area of $\Delta ABC = \frac{1}{2}\cdot 2\cdot 3=3$.

2. Find the partial fraction decomposition of $\frac{x^3}{(x-2)(x+1)}$. (8 marks)

Ans.: Performing long division enables us to write $\frac{x^3}{(x-2)(x+1)} = x+1+\frac{3x+2}{(x-2)(x+1)}$.

The partial fraction decomposition of $\frac{3x+2}{(x-2)(x+1)}$ has the form $\frac{3x+2}{(x-2)(x+1)} = \frac{A}{x-2}+\frac{B}{x+1}$.

Comparing terms on both sides yields $A=\frac{8}{3}$ and $B=\frac{1}{3}$.

$\therefore \frac{x^3}{(x-2)(x+1)} = x+1+\frac{\frac{8}{3}}{x-2}+\frac{\frac{1}{3}}{x+1} = x+1+\frac{8}{3(x-2)}+\frac{1}{3(x+1)}$.

3. In Bag A there are 4 red balls, 6 yellow balls and 10 green balls. In Bag B there are 12 red balls, 4 yellow balls and 4 green balls. If one ball is randomly selected from each bag, find the probability that
   (a) both balls are red. (2 marks)
   (b) both balls are not red. (2 marks)
   (c) at least one ball is red. (2 marks)
   (d) both balls are red or both balls are not red. (2 marks)

Ans.: (a) The probabilities of drawing a red ball from Bag A and Bag B are respectively $\frac{4}{20}$ and $\frac{12}{20}$. The event of drawing a ball from Bag A is clearly independent of the event of drawing a ball from Bag B.

$\therefore$ The required probability of this part $=\frac{4}{20}\cdot\frac{12}{20}=\frac{1}{5}\cdot\frac{3}{5}=\frac{3}{25}$.

(b) From (a), the required probability of this part $=\left(1-\frac{1}{5}\right)\cdot\left(1-\frac{3}{5}\right)=\frac{8}{25}$.

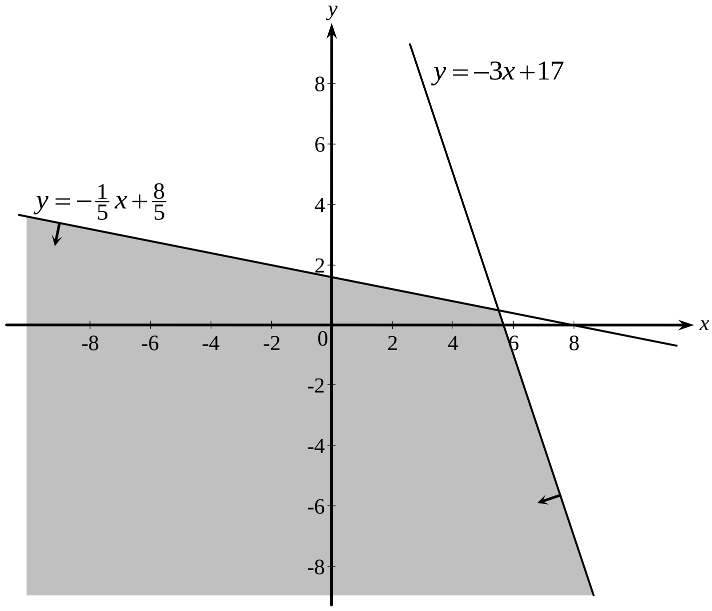
(c) From (b), the required probability of this part $=1-\frac{8}{25}=\frac{17}{25}$.

(d) Since the events in (a) and (b) are mutually exclusive, the required probability of this part $=\frac{3}{25}+\frac{8}{25}=\frac{11}{25}$.

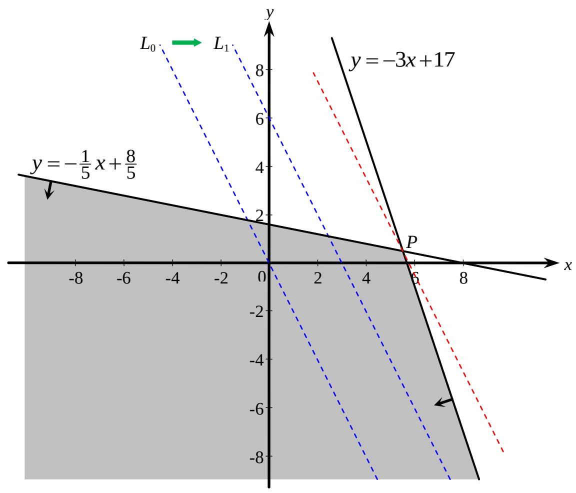
4. (a) Solve the system of inequalities $\begin{cases} x+5y \le 8 \\ 3x+y \le 17 \end{cases}$ graphically. (4 marks)

(b) Let $C=2x+y$, where (x, y) is a solution of the system of inequalities in (a). What is the maximum value of C? (4 marks)

Ans.: (a) The graphical solution of the given inequalities is the common region of the lower half-plane of $y=-\frac{1}{5}x+\frac{8}{5}$ and the lower half-plane of $y=-3x+17$ (including the boundary), which is shown as the shaded area in the picture below.



(b) Let L_0 and L_1 denote respectively the lines $2x+y=0$ and $2x+y=6$. By considering the positions of these 2 lines, we see that parallel shifts in the line L_0 to the right produce bigger values of C, therefore, we have to move L_0 to the right as far as possible while L_0 remains touching the shaded region.



The above picture shows that the value of C is maximized at the point P, which is the intersection of $y=-\frac{1}{5}x+\frac{8}{5}$ and $y=-3x+17$. Solving these equations, we get $P=\left(\frac{11}{2}, \frac{1}{2}\right)$.

$\therefore$ Maximum value of $C=2\cdot\frac{11}{2}+\frac{1}{2}=\frac{23}{2}=11.5$.

5. (a) Let x and y be integers. Prove that if both $x-y$ and y are divisible by 6, so is x. (2 marks)

(b) Use (a) and mathematical induction to show that $4n^3+3n^2+5n$ is a multiple of 6 for all positive integers n. (6 marks)

Ans.: (a) If both $x-y$ and y are divisible by 6, then $x-y=6m$ and $y=6n$ for some integers m and n.

$\therefore\ x=(x-y)+y=6m+6n=6(m+n)$, which is a multiple of 6. This completes the proof.

(b) Let $S(n)$ denote the statement "$4n^3+3n^2+5n$ is a multiple of 6".

(i) $S(1)$ is true $\because\ 4\cdot 1^3+3\cdot 1^2+5\cdot 1=12$, which is a multiple of 6.

(ii) Assume $S(k)$ is true for some positive integer k, that is

$4k^3+3k^2+5k$ is divisible by 6 ---------------- (1)

From the following equalities

$$\begin{aligned}&[4(k+1)^3+3(k+1)^2+5(k+1)]-(4k^3+3k^2+5k)\\ &=4(k^3+3k^2+3k+1)+3(k^2+2k+1)+5(k+1)-(4k^3+3k^2+5k)\\ &=6(2k^2+3k+2)\end{aligned}$$

we see that

$[4(k+1)^3+3(k+1)^2+5(k+1)]-(4k^3+3k^2+5k)$ is divisible by 6 ---------------- (2)

Using (2), (1) and (a), we conclude that $4(k+1)^3+3(k+1)^2+5(k+1)$ is divisible by 6.

This means that $S(k+1)$ is also true.

By (i), (ii), and the Principle of Mathematical Induction, $S(n)$ is true for all positive integers n.