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Macao Institute for Tourism Studies

澳門科技大學
UNIVERSIDADE DE CIÊNCIA E TECNOLOGIA DE MACAU
MACAU UNIVERSITY OF SCIENCE AND TECHNOLOGY

澳門四高校聯合入學考試（語言科及數學科）

Joint Admission Examination for Macao Four Higher Education Institutions (Languages and Mathematics)

2019 年試題及參考答案

2019 Examination Paper and Suggested Answer

數學附加卷　Mathematics Supplementary Paper

注意事項：

1. 考生獲發文件如下:

   1.1 本考卷包括封面共 22 版

   1.2 草稿紙一張

2. 請於本考卷封面填寫聯考編號、考場、樓宇、考室及座號。
3. 本考卷共有五條解答題，每題二十分，任擇三題作答。全卷滿分為六十分。
4. 必須在考卷內提供的橫間頁內作答，寫在其他地方的答案將不會獲評分。
5. 必須將解題步驟清楚寫出。只當答案和所有步驟正確而清楚地表示出來，考生方可獲得滿分。
6. 本考卷的圖形並非按比例繪畫。
7. 考試中不可使用任何形式的計算機。
8. 請用藍色或黑色原子筆作答。
9. 考試完畢，考生須交回本考卷及草稿紙。

Instructions:

1. Each candidate is provided with the following documents:

   1.1 Question paper including cover page – 22 pages

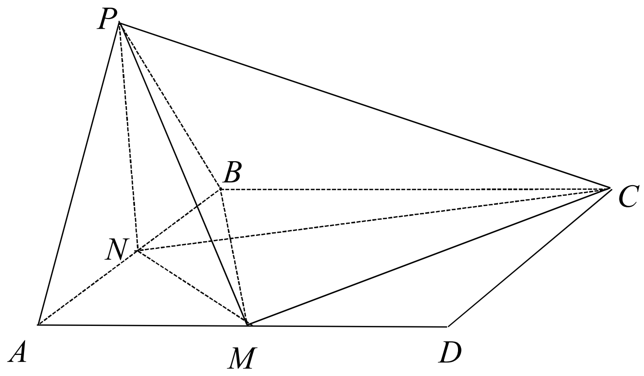
   1.2 One sheet of draft paper

2. Fill in your JAE No., campus, building, room and seat no. on the front page of the examination paper.
3. There are 5 questions in this paper, each carries 20 marks. Answer any 3 questions. Full mark of this paper is 60.
4. Put your answers in the lined pages provided. Answers put elsewhere will not be marked.
5. Show all your steps in getting to the answer. Full credits will be given only if the answer and all the steps are correct and clearly shown.
6. The diagrams in this examination paper are not drawn to scale.
7. Calculators of any kind are not allowed in the examination.
8. Answer the questions with a blue or black ball pen.
9. Candidates must return the question paper and draft paper at the end of the examination.

任擇三題作答，每題二十分。請把答案寫在緊接每條題目之後的 3 頁橫間頁內。

Answer any 3 questions, each carries 20 marks. Write down the answers on the 3 lined pages following each question.

1.



如上圖所示，PAB 是邊長為 1 的等邊三角形，$ABCD$ 是一長方形，

$|BC| = \sqrt{2}$ ，平面 PAB 與平面 ABC 垂直，M 和 N 分別為 AD 和 AB 的中點。

(a) (i) 證明 PN 垂直平面 ABC，從而求三棱錐 P-BCM 的體積。 (5 分)

(ii) 求三角形 PBM 的面積，從而求點 C 至平面 PBM 的距離。 (6 分)

(b) (i) 證明 $\angle NMC$ 是一直角。[提示: 考慮三角形 NMC 的邊長。] (4 分)

(ii) 求二面角 $N-MC-P$ 。[提示: 證明 $PM \perp MC$ 。] (5 分)

In the above figure, PAB is an equilateral triangle with edge length 1, $ABCD$ is a rectangle, $|BC| = \sqrt{2}$, plane PAB and plane ABC are perpendicular, M and N are the midpoints of AD and AB, respectively.

(a) (i) Show that PN is perpendicular to the plane ABC. Hence find the volume of the triangular pyramid P-BCM. (5 marks)

(ii) Find the area of the triangle PBM. Hence find the distance from point C to the plane PBM. (6 marks)

(b) (i) Show that $\angle NMC$ is a right angle. [Hint. Consider the lengths of the sides of triangle NMC .] (4 marks)

(ii) Find the dihedral angle $N-MC-P$. [Hint. Show that $PM \perp MC$.] (5 marks)

2. (a) 設曲線 $y = f(x)$ 通過點 (0,3) 及 $f'(x) = x^2 - 2x - 3$。

(i) 求 $f(x)$ 及 $f''(x)$。 (3 分)

(ii) 求 $f(x)$ 的局部極大點、局部極小點和拐點。 (5 分)

(iii) 繪出曲線 $y = f(x)$，$-2 \le x \le 4$。 (3 分)

(iv) 繪出曲線 $y = |f(-x)|$，$-4 \le x \le 2$。 (1 分)

(b) 求由曲線 $y = -x^2 + 4x + 3$ 及 $y = 3|x-1|$ 所包圍的區域的面積。 (8 分)

(a) Suppose the curve $y = f(x)$ passes through the point (0,3) and $f'(x) = x^2 - 2x - 3$.

(i) Find $f(x)$ and $f''(x)$. (3 marks)

(ii) Find the local maximum points, local minimum points and inflection points of $f(x)$. (5 marks)

(iii) Sketch the curve $y = f(x)$, $-2 \le x \le 4$. (3 marks)

(iv) Sketch the curve $y = |f(-x)|$, $-4 \le x \le 2$. (1 marks)

(b) Find the area of the region bounded by the curves $y = -x^2 + 4x + 3$ and $y = 3|x-1|$. (8 marks)

3. 已知拋物線$P: y^2 = x$及點$A(t^2, t)$ $(t \neq 0)$ 是在拋物線上。

(a) (i) 證明直線 $y = mx + c$是 P 的一條切線當且僅當 $4mc = 1$。 (5 分)

(ii) 推導出 P 在點A 的切線的斜率為 $\frac{1}{2t}$。 (3 分)

(b) (i) 給出 P 在點A 的法線的方程。 (2 分)

(ii) 證明P有兩條非水平的法線通過 $B(h,0)$ 當且僅當 $h > 1/2$。 (4 分)

(iii) 設 P 有兩條非水平法線通過 $B(h,0)$，且它們的夾角是$\frac{\pi}{4}$，求 h 的值。 (6 分)

Given parabola $P: y^2 = x$ and that point $A(t^2, t)$ $(t \neq 0)$ is on the parabola.

(a) (i) Show that the line $y = mx + c$ is a tangent line of *P* if and only if $4mc = 1$. (5 marks)

(ii) Deduce that the slope of the tangent line of *P* at point *A* is $\frac{1}{2t}$. (3 marks)

(b) (i) Give an equation of normal line of *P* at point *A*. (2 marks)

(ii) Show that there are two non-horizontal normal lines of *P* passing through $B(h,0)$ if and only if $h > 1/2$. (4 marks)

(iii) Suppose there are two non-horizontal normal lines of *P* passing through $B(h,0)$ and the angle between them is $\frac{\pi}{4}$, find the value of *h*. (6 marks)

4. 設 $i=\sqrt{-1}$ 。

(a) 設複數 $z=\dfrac{1+i}{1+\sqrt{3}i}$ 。

(i) 按極式 $r(\cos\alpha+i\sin\alpha)$ 表示 z，其中 $-\pi<\alpha\leq\pi$ 。 (4 分)

(ii) 求 z^{2019} 。 (4 分)

(b) (i) 用棣美弗定理，證明

$$\cos 7\alpha = 64\cos^7\alpha - 112\cos^5\alpha + 56\cos^3\alpha - 7\cos\alpha$$ 。 (6 分)

(ii) 用 (i) 的結果及代換 $x=4\cos^2\alpha$，解方程 $x^3-7x^2+14x-7=0$ 。 (6 分)

Let $i=\sqrt{-1}$.

(a) Let complex number $z=\dfrac{1+i}{1+\sqrt{3}i}$.

(i) Express z in polar form $r(\cos\alpha+i\sin\alpha)$, $-\pi<\alpha\leq\pi$. (4 marks)

(ii) Find z^{2019}. (4 marks)

(b) (i) Using De Moivre's theorem, show that

$$\cos 7\alpha = 64\cos^7\alpha - 112\cos^5\alpha + 56\cos^3\alpha - 7\cos\alpha.$$ (6 marks)

(ii) Using the result in (i) and the substitution $x=4\cos^2\alpha$, solve the equation

$$x^3-7x^2+14x-7=0.$$ (6 marks)

5. (a) 因式分解行列式 $\begin{vmatrix} 1 & a^3 & bc \\ 1 & b^3 & ac \\ 1 & c^3 & ab \end{vmatrix}$。 (8 分)

(b) 設 p 和 q 為常數。已知以 x, y, z 為未知量的方程組 (E): $\begin{cases} x-2y+pz=q \\ -x+\ \ y+2z=1 \\ -x+py-2z=0 \end{cases}$。

(i) 求 p 的取值範圍使得 (E) 有唯一解。 (4 分)

(ii) 對 $p=0$ 及 $q=4$，求 (E) 的解。 (4 分)

(iii) 對 $p=-3$，求 q 的值使得 (E) 有解，並求 (E) 的解。 (4 分)

(a) Factorize the determinant $\begin{vmatrix} 1 & a^3 & bc \\ 1 & b^3 & ac \\ 1 & c^3 & ab \end{vmatrix}$. (8 marks)

(b) Let p and q be constants. Given, in unknowns x, y, z the system of equations

$$(E):\ \begin{cases} x-2y+pz=q \\ -x+\ \ y+2z=1 \\ -x+py-2z=0 \end{cases}.$$

(i) Find the range of p such that (E) has a unique solution. (4 marks)

(ii) Solve (E) when $p=0$ and $q=4$. (4 marks)

(iii) For $p=-3$, find the value of q such that (E) has a solution, and solve (E). (4 marks)

參考答案：

1. (a) (i) 因 ΔPAB 是等邊的，而 N 是 AB 的中點，故 $PN \perp AB$。已知平面 PAB 與平面 ABC 垂直，故 PN 垂直平面 ABC。因 $|PN| = |AP| \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ 及

ΔBCM 的面積 $= \frac{1}{2}|AB||BC| = \frac{\sqrt{2}}{2}$，故 $P-BCM$ 的體積是 $\frac{\sqrt{6}}{12}$。

(ii) 從 $|BM| = |PM| = \sqrt{\frac{3}{2}}$，得知 ΔPMB 是等腰的。已知 $|PB| = 1$，計算得出

ΔPMB 的面積是 $\frac{\sqrt{5}}{4}$。設 h 為點 C 至平面 PBM 的距離，則

$C-PBM$ 的體積是 $\frac{\sqrt{5}h}{12}$。結合 (i)的結果 ，得 $h = \sqrt{\frac{6}{5}}$。

(b) (i) 計算得出 $|MN|^2 = \frac{3}{4}$，$|MC|^2 = \frac{3}{2}$，$|NC|^2 = \frac{9}{4}$。因 $|MN|^2 + |MC|^2 = |NC|^2$，

故 $\angle NMC$ 是一直角。

(ii) 因 PN 垂直平面 ABC，PM 在平面 ABC 的射影為 NM。

故 $\angle PMC = \angle NMC = \frac{\pi}{2}$。

二面角 $N-MC-P$ $=$ $\angle PMN = \tan^{-1}\frac{|PN|}{|MN|} = \tan^{-1}\frac{\sqrt{3}/2}{\sqrt{3}/2} = \frac{\pi}{4}$。

2. (a) (i) $f(x) = \int x^2 - 2x - 3\,dx = \frac{x^3}{3} - x^2 - 3x + C$，$f(0) = 3 \Rightarrow C = 3$。$f''(x) = 2x - 2$。

(ii) $f'(x) = 0 \Leftrightarrow x^2 - 2x - 3 = 0 \Leftrightarrow (x-3)(x+1) = 0 \Leftrightarrow x = 3$ or $x = -1$。

當 $x < -1$ 時，$f'(x) > 0$， 故 $f(x)$ 是遞增的。

當 $-1 < x < 3$ 時，$f'(x) < 0$， 故 $f(x)$ 是遞減的。

當 $3 < x$ 時，$f'(x) > 0$， 故 $f(x)$ 是遞增的。

因此，$f(-1) = \frac{14}{3}$ 是一局部極大點及 $f(3) = -6$ 是一局部極小點。

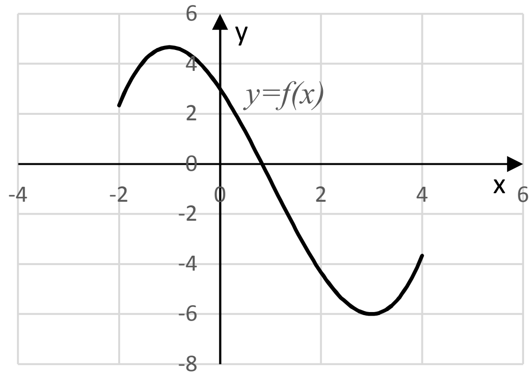
$f''(x) = 0 \Leftrightarrow 2x - 2 = 0 \Leftrightarrow x = 1$。

當 $x < 1$ 時，$f''(x) < 0$，故 $f(x)$ 是凹的。

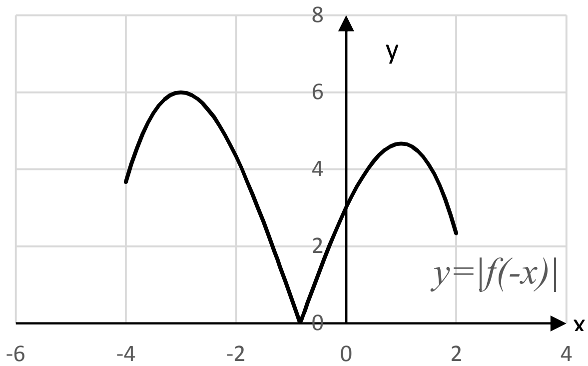
當 $x > 1$ 時，$f''(x) > 0$，故 $f(x)$ 是凸的。

因此，$f(1) = -\frac{2}{3}$ 是一拐點。

(iii)



(iv)



(b) 當 $x \geq 1$，解 $\begin{cases} y = -x^2 + 4x + 3 \\ y = 3x - 3 \end{cases}$，得 $x=3$。

當 $x \leq 1$，解 $\begin{cases} y = -x^2 + 4x + 3 \\ y = 3 - 3x \end{cases}$，得 $x=0$。

當 $0 \leq x \leq 3$，曲線 $y=3|x-1|$ 是在曲線 $y=-x^2+4x+3$ 之下，故所求面積為

$$\int_0^1 (-x^2+4x+3)-(3-3x)\ dx + \int_1^3 (-x^2+4x+3)-(3x-3)\ dx$$
$$= \int_0^1 -x^2+7x\ dx + \int_1^3 -x^2+x+6\ dx$$
$$= \left[-\frac{x^3}{3}+\frac{7}{2}x^2\right]_0^1 + \left[-\frac{x^3}{3}+\frac{x^2}{2}+6x\right]_1^3$$
$$= \frac{21}{2}$$

3. (a) (i) 由 $\begin{cases} y^2 = x \\ y = mx + c \end{cases}$，得

$$m^2x^2 + (2mc-1)x + c^2 = 0 \text{ 。} \qquad (*)$$

因 $y = mx + c$ 是 P 的切線，故 (*) 有二重根，其判別式等於 0。因此得

$(2mc-1)^2 - 4m^2c^2 = 0$，即 $4mc = 1$。

(ii) 從 (i) 得切線的方程為 $y = mx + \dfrac{1}{4m}$。因切線通過 $A(t^2, t)$，故得出

$t = mt^2 + \dfrac{1}{4m} \Rightarrow 4m^2t^2 - 4mt + 1 = 0 \Rightarrow (2mt-1)^2 = 0 \Rightarrow 2mt = 1 \Rightarrow m = \dfrac{1}{2t}$。

(b) (i) $y = -2t(x - t^2) + t$

(ii) 法線通過 $B(h, 0)$ 當且僅當 $0 = -2t(h - t^2) + t$。因 $t \neq 0$，有 $2h - 1 = 2t^2$。

因 $2t^2 > 0$，故得出法線通過 $B(h, 0)$ 當且僅當 $h > \dfrac{1}{2}$。

當 $h > \dfrac{1}{2}$，$t = \pm\sqrt{h - \dfrac{1}{2}}$，故有兩條法線。

(c) 由 (b) (ii) 知 $t = \pm\sqrt{h - \dfrac{1}{2}}$，故通過 $B(h, 0)$ 的兩條法線的斜率 m_1, m_2 為

$\pm\sqrt{4h-2}$。設 α 為這兩條法線的夾角，則

$$1 = \tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \left|\frac{2\sqrt{4h-2}}{1-(4h-2)}\right| = \left|\frac{2\sqrt{4h-2}}{3-4h}\right| \text{ 。}$$

由此得到 $(3-4h)^2 - 4(4h-2) = 0$，即 $16h^2 - 40h + 17 = 0$。故 $h = \dfrac{5 \pm \sqrt{8}}{4}$。

4. (a)(i)

$$z=\frac{1+i}{1+\sqrt{3}i}=\frac{\sqrt{2}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})}{2(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})}=\frac{1}{\sqrt{2}}\left(\cos(\frac{\pi}{4}-\frac{\pi}{3})+i\sin(\frac{\pi}{4}-\frac{\pi}{3})\right)$$

$$=\frac{1}{\sqrt{2}}\left(\cos(-\frac{\pi}{12})+i\sin(-\frac{\pi}{12})\right)$$

(ii)

$$z^{2019}=\frac{1}{2^{2019/2}}\left(\cos(-\frac{2019\pi}{12})+i\sin(-\frac{2019\pi}{12})\right)$$

$$=\frac{1}{2^{2019/2}}\left(\cos(-168\pi-\frac{3\pi}{12})+i\sin(-168\pi-\frac{3\pi}{12})\right)=\frac{1}{2^{2019/2}}\left(\cos(-\frac{\pi}{4})+i\sin(-\frac{\pi}{4})\right)$$

$$=\frac{1}{2^{2019/2}}\left(\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)=\frac{1}{2^{1010}}(1-i)$$

(b)(i)

$$\cos 7\alpha=\text{Re}[(\cos\alpha+i\sin\alpha)^7]$$

$$=\cos^7\alpha-C_7^5\cos^5\alpha\sin^2\alpha+C_7^3\cos^3\alpha\sin^4\alpha-C_7^1\cos\alpha\sin^6\alpha$$

$$=\cos^7\alpha-21\cos^5\alpha(1-\cos^2\alpha)+35\cos^3\alpha(1-\cos^2\alpha)^2-7\cos\alpha(1-\cos^2\alpha)^3$$

$$=64\cos^7\alpha-112\cos^5\alpha+56\cos^3\alpha-7\cos\alpha$$

(ii) 用代換 $x=4\cos^2\alpha$ ，得 $64\cos^6\alpha-112\cos^4\alpha+56\cos^2\alpha-7=0$ 。

全式乘 $\cos\alpha$ ，得 $64\cos^7\alpha-112\cos^5\alpha+56\cos^3\alpha-7\cos\alpha=0$ 。

故現須在 $\cos\alpha\neq 0$ 及 $0\leq\alpha<2\pi$ 條件下解 $\cos 7\alpha=0$ ，而其解為

$\alpha=\frac{\pi}{14}+\frac{k\pi}{7}$ ，$k=0,1,2,4,5,6$ 。因 $\cos^2\frac{\pi}{14}=\cos^2\frac{13\pi}{14}$ ，$\cos^2\frac{3\pi}{14}=\cos^2\frac{11\pi}{14}$ 及

$\cos^2\frac{5\pi}{14}=\cos^2\frac{9\pi}{14}$ ，故方程的解為 $4\cos^2\frac{\pi}{14}$ ，$4\cos^2\frac{3\pi}{14}$ 和 $4\cos^2\frac{5\pi}{14}$ 。

5. (a)

$$\begin{vmatrix}1 & a^3 & bc\\1 & b^3 & ac\\1 & c^3 & ab\end{vmatrix}=\begin{vmatrix}1 & a^3 & bc\\0 & b^3-a^3 & c(a-b)\\0 & c^3-a^3 & b(a-c)\end{vmatrix}=(b-a)(c-a)\begin{vmatrix}1 & a^3 & bc\\0 & b^2+ab+a^2 & -c\\0 & c^2+ac+a^2 & -b\end{vmatrix}$$

$$=(b-a)(c-a)\begin{vmatrix}1 & a^3 & bc\\0 & b^2+ab+a^2 & -c\\0 & c^2-b^2+a(c-b) & c-b\end{vmatrix}=(b-a)(c-a)(c-b)\begin{vmatrix}1 & a^3 & bc\\0 & b^2+ab+a^2 & -c\\0 & c+b+a & 1\end{vmatrix}$$

$$=(a-b)(b-c)(c-a)(a^2+b^2+c^2+ab+ac+bc)$$

(b) (i) (E) 有唯一解的條件是 $\begin{vmatrix}1 & -2 & p\\-1 & 1 & 2\\-1 & p & -2\end{vmatrix}\neq 0$，$p$ 的取值範圍是

$\{p: p\neq -3$ 及 $p\neq 2\}$。

(ii) $x=-2$，$y=-3$，$z=1$。

(iii) 設 $p=-3$，從 (E) 得 $\begin{cases}x-2y-3z=q\\ \quad -y\ \ -z=1+q\\ \quad -5y-5z=q\end{cases}$。從第二及第三條方程得 $q=-\dfrac{5}{4}$ 。

解 $\begin{cases}x-2y-3z=-\dfrac{5}{4}\\ \quad -y\ \ -z=-\dfrac{1}{4}\end{cases}$，得 $x=t-\dfrac{3}{4}$，$y=\dfrac{1}{4}-t$，$z=t$，其中 t 為實數。

Suggested Answer

1. (a) (i) Since ΔPAB is equilateral and N is the midpoint of AB, we have $PN \perp AB$. Given the planes PAB and ABC are perpendicular, hence PN is perpendicular to plane ABC. As $|PN| = |AP|\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and

the area of $\Delta BCM = \frac{1}{2}|AB||BC| = \frac{\sqrt{2}}{2}$, the volume of $P-BCM$ is $\frac{\sqrt{6}}{12}$.

(ii) From $|BM| = |PM| = \sqrt{\frac{3}{2}}$, ΔPMB is isosceles. Given $|PB| = 1$, by direct calculation, the area of ΔPMB is $\frac{\sqrt{5}}{4}$. Let h be the distance from point C to the plane PBM. Then, the volume of $C-PBM$ is $\frac{\sqrt{5}h}{12}$. Together with the result in (i), we get $h = \sqrt{\frac{6}{5}}$.

(b) (i) By direct calculation, $|MN|^2 = \frac{3}{4}$, $|MC|^2 = \frac{3}{2}$, $|NC|^2 = \frac{9}{4}$. Since $|MN|^2 + |MC|^2 = |NC|^2$, $\angle NMC$ is a right angle.

(ii) Since PN is perpendicular to the plane ABC, the projection of PM on the plane ABC is NM. Hence $\angle PMC = \angle NMC = \frac{\pi}{2}$.

dihedral angle $N-MC-P = \angle PMN = \tan^{-1}\frac{|PN|}{|MN|} = \tan^{-1}\frac{\sqrt{3}/2}{\sqrt{3}/2} = \frac{\pi}{4}$.

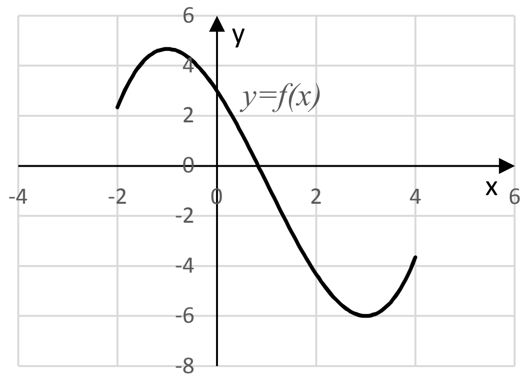
2. (a) (i) $f(x) = \int x^2 - 2x - 3\, dx = \frac{x^3}{3} - x^2 - 3x + C$, $f(0) = 3 \Rightarrow C = 3$. $f''(x) = 2x - 2$.

(ii) $f'(x) = 0 \Leftrightarrow x^2 - 2x - 3 = 0 \Leftrightarrow (x-3)(x+1) = 0 \Leftrightarrow x = 3$ or $x = -1$.
When $x < -1$, $f'(x) > 0$. Hence $f(x)$ is increasing.
When $-1 < x < 3$, $f'(x) < 0$. Hence $f(x)$ is decreasing.
When $3 < x$, $f'(x) > 0$. Hence $f(x)$ is increasing.
Consequently, $f(-1) = \frac{14}{3}$ is a local maximum point and $f(3) = -6$ is a local minimum point.

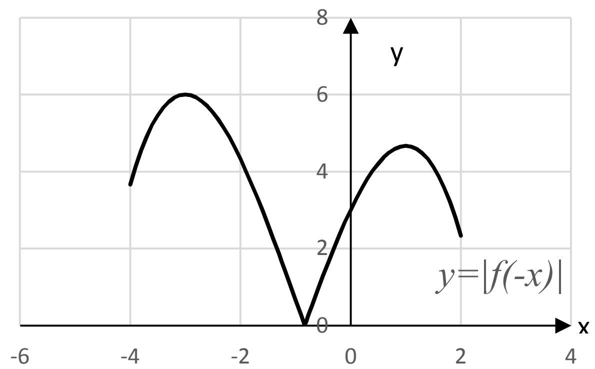
$f''(x)=0 \Leftrightarrow 2x-2=0 \Leftrightarrow x=1$.

When $x<1$, $f''(x)<0$. Hence $f(x)$ is concave. When $x>1$, $f''(x)>0$. Hence $f(x)$ is convex. Hence, $f(1)=-\frac{2}{3}$ is an inflection point.

(iii)



(iv)



(b) When $x\geq 1$, solving $\begin{cases} y=-x^2+4x+3 \\ y=3x-3 \end{cases}$, we get $x=3$.

When $x\leq 1$, solving $\begin{cases} y=-x^2+4x+3 \\ y=3-3x \end{cases}$, we get $x=0$.

When $0\leq x\leq 3$, the curve $y=3|x-1|$ is below the curve $y=-x^2+4x+3$. Hence the required area is

$$\int_0^1 (-x^2+4x+3)-(3-3x)\ dx+\int_1^3 (-x^2+4x+3)-(3x-3)\ dx$$

$$=\int_0^1 -x^2+7x\ dx+\int_1^3 -x^2+x+6\ dx$$

$$=\left[-\frac{x^3}{3}+\frac{7}{2}x^2\right]_0^1+\left[-\frac{x^3}{3}+\frac{x^2}{2}+6x\right]_1^3$$

$$=\frac{21}{2}$$

3. (a) (i) From $\begin{cases} y^2 = x \\ y = mx + c \end{cases}$, we get

$$m^2x^2 + (2mc - 1)x + c^2 = 0 . \qquad (*)$$

As $y = mx + c$ is a tangent line of P, (*) has a double root and hence its discriminant is 0. Thus $(2mc-1)^2 - 4m^2c^2 = 0$, i.e. $4mc = 1$.

(ii) From (i) an equation of the tangent line is $y = mx + \dfrac{1}{4m}$. As the tangent line passes through $A(t^2, t)$, we get

$$t = mt^2 + \frac{1}{4m} \Rightarrow 4m^2t^2 - 4mt + 1 = 0 \Rightarrow (2mt-1)^2 = 0 \Rightarrow 2mt = 1 \Rightarrow m = \frac{1}{2t}.$$

(b) (i) $y = -2t(x - t^2) + t$

(ii) The normal line passes through $B(h,0)$ if and only if $0 = -2t(h - t^2) + t$.

As $t \neq 0$, we have $2h - 1 = 2t^2$.

As $2t^2 > 0$, the normal line passes through $B(h,0)$ if and only if $h > \dfrac{1}{2}$.

When $h > \dfrac{1}{2}$, $t = \pm\sqrt{h - \dfrac{1}{2}}$ and hence there are two normal lines.

(c) From (b) (ii), $t = \pm\sqrt{h - \dfrac{1}{2}}$ and so the slopes m_1, m_2 of the two normal lines passing through $B(h,0)$ are $\pm\sqrt{4h-2}$. Let α be the angle between the two normal lines. Then

$$1 = \tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \left|\frac{2\sqrt{4h-2}}{1-(4h-2)}\right| = \left|\frac{2\sqrt{4h-2}}{3-4h}\right|.$$

Hence, $(3-4h)^2 - 4(4h-2) = 0$, i.e. $16h^2 - 40h + 17 = 0$. Thus $h = \dfrac{5 \pm \sqrt{8}}{4}$.

4. (a)(i)

$$z = \frac{1+i}{1+\sqrt{3}i} = \frac{\sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})}{2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})} = \frac{1}{\sqrt{2}}\left(\cos(\frac{\pi}{4} - \frac{\pi}{3}) + i\sin(\frac{\pi}{4} - \frac{\pi}{3})\right)$$

$$= \frac{1}{\sqrt{2}}\left(\cos(-\frac{\pi}{12}) + i\sin(-\frac{\pi}{12})\right)$$

(ii)

$$z^{2019} = \frac{1}{2^{2019/2}}\left(\cos(-\frac{2019\pi}{12}) + i\sin(-\frac{2019\pi}{12})\right)$$

$$= \frac{1}{2^{2019/2}}\left(\cos(-168\pi - \frac{3\pi}{12}) + i\sin(-168\pi - \frac{3\pi}{12})\right) = \frac{1}{2^{2019/2}}\left(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})\right)$$

$$= \frac{1}{2^{2019/2}}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \frac{1}{2^{1010}}(1-i)$$

(b) (i)

$$\cos 7\alpha = \mathrm{Re}[(\cos\alpha + i\sin\alpha)^7]$$
$$= \cos^7\alpha - C_7^5\cos^5\alpha\sin^2\alpha + C_7^3\cos^3\alpha\sin^4\alpha - C_7^1\cos\alpha\sin^6\alpha$$
$$= \cos^7\alpha - 21\cos^5\alpha(1-\cos^2\alpha) + 35\cos^3\alpha(1-\cos^2\alpha)^2 - 7\cos\alpha(1-\cos^2\alpha)^3$$
$$= 64\cos^7\alpha - 112\cos^5\alpha + 56\cos^3\alpha - 7\cos\alpha$$

(ii) Using the substitution $x = 4\cos^2\alpha$, we get $64\cos^6\alpha - 112\cos^4\alpha + 56\cos^2\alpha - 7 = 0$.
Multiplying it by $\cos\alpha$, we get $64\cos^7\alpha - 112\cos^5\alpha + 56\cos^3\alpha - 7\cos\alpha = 0$.
Thus, we need to solve $\cos 7\alpha = 0$ under the conditions $\cos\alpha \neq 0$ and $0 \le \alpha < 2\pi$.
The solutions are $\alpha = \dfrac{\pi}{14} + \dfrac{k\pi}{7}$, $k = 0,1,2,4,5,6$.
As $\cos^2\dfrac{\pi}{14} = \cos^2\dfrac{13\pi}{14}$, $\cos^2\dfrac{3\pi}{14} = \cos^2\dfrac{11\pi}{14}$ and $\cos^2\dfrac{5\pi}{14} = \cos^2\dfrac{9\pi}{14}$, the solutions of the given equation are $4\cos^2\dfrac{\pi}{14}$, $4\cos^2\dfrac{3\pi}{14}$ and $4\cos^2\dfrac{5\pi}{14}$.

5. (a)

$$\begin{vmatrix} 1 & a^3 & bc \\ 1 & b^3 & ac \\ 1 & c^3 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^3 & bc \\ 0 & b^3-a^3 & c(a-b) \\ 0 & c^3-a^3 & b(a-c) \end{vmatrix} = (b-a)(c-a)\begin{vmatrix} 1 & a^3 & bc \\ 0 & b^2+ab+a^2 & -c \\ 0 & c^2+ac+a^2 & -b \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & a^3 & bc \\ 0 & b^2+ab+a^2 & -c \\ 0 & c^2-b^2+a(c-b) & c-b \end{vmatrix} = (b-a)(c-a)(c-b)\begin{vmatrix} 1 & a^3 & bc \\ 0 & b^2+ab+a^2 & -c \\ 0 & c+b+a & 1 \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)(a^2+b^2+c^2+ab+ac+bc)$$

(b) (i) The condition that (E) has a unique solution is $\begin{vmatrix} 1 & -2 & p \\ -1 & 1 & 2 \\ -1 & p & -2 \end{vmatrix} \neq 0$.

The range of p is $\{p : p \neq -3 \text{ and } p \neq 2\}$.

(ii) $x = -2$，$y = -3$，$z = 1$.

(iii) Let $p = -3$. From (E) we get $\begin{cases} x - 2y - 3z = q \\ \quad -y \ \ - z = 1 + q \\ -5y - 5z = q \end{cases}$. From the second and third

equations we get $q = -\dfrac{5}{4}$. Solving $\begin{cases} x - 2y - 3z = -\dfrac{5}{4} \\ \quad -y \ \ - z = -\dfrac{1}{4} \end{cases}$, we get $x = t - \dfrac{3}{4}$, $y = \dfrac{1}{4} - t$,

$z = t$, where t is a real number.